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LIE THEORY OF FINITE SIMPLE GROUPS AND THE GENERALISED ROTH CONJECTURE

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Abstract. We apply a recent approach to noncommutative differential geometry on finite groups to the case of finite nonabelian simple groups and similar groups with trivial centre. The 'Lie algebra' or bicovariant differential calculus here is provided by an ad-stable generating set and by analogy with Lie theory we consider when the associated Killing form is nondegenerate. We prove nondegeneracy for the case of the universal calculus whenever Roth's property holds, including for all symmetric groups S_n, all sporadic and most other finite simple nonabelian groups. We conjecture that nondegeneracy holds more generally and prove it for the 2-cycles calculus on any S_n, and by computer for all real conjugacy classes on finite simple nonabelian groups up to order 75,000. In all cases we find that the Killing form is in fact either positive definite, if the conjugacy class consists of involutions, or otherwise has zero (evenly split) signature. As an application of the Killing form we find that its eigenspaces typically decompose the conjugacy class representation into irreducibles and we explore the possibility of bijectively assigning an irreducible representation to a conjugacy class containing it. We prove that the conjugacy classes of S_n containing the sign representation are those corresponding to partitions of n into distinct odd parts and relate this observation to a classical identity of Euler.

1. Introduction

Roth's conjecture [15] in the theory of finite groups asserts that the ad-representation of the group G on itself contains every complex irreducible representation of $G/Z(G)$ at least once (where $Z(G)$ is the centre). This conjecture is false in general, but is known to be true for symmetric groups [3] and alternating groups [17], and, recently, for the sporadic simple groups [7] using methods from [13]. We shall refer to this property, when it holds, as the Roth property. Indeed, for simple groups the exceptions according to [7] appear to be rather few and to amount to some instances of one classical family of Lie type over finite fields of particular order. In this paper we provide a noncommutative-geometrical point of view on this property, relating it to the universal differential calculus on the finite group and nondegeneracy of the corresponding Killing form (these notions will be defined in the preliminaries

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section of the paper). Nondegeneracy of the Killing form here appears to be weaker than the Roth property and we conjecture that it holds at least for all simple G.

Beyond this, we expect more strongly at least when G is simple or S_n that the Killing form is nondegenerate quite generally, not only for the universal calculus. A general calculus on a finite group corresponds to an ad-stable subset $\mathcal{C} \subseteq G \setminus \{e\}$ with equality in the case of the universal calculus. Here e is the group identity. The Killing form is defined on $\mathcal{L} = \mathbb{C}\mathcal{C}$ and in the case of finite groups reduces to the trace in the representation

$$K(x_a, x_b) := \chi_{\mathcal{L}}(ab)$$

where $\{x_a \mid a \in \mathcal{C}\}$ is a basis of $\mathcal{L}$ and $\chi_{\mathcal{L}}$ is the character or trace in the representation $\mathcal{L}$. In our case the representation $\mathcal{L}$ is given by the adjoint action hence

$$K(x_a, x_b) = |Z(ab) \cap \mathcal{C}|$$

where $Z(g)$ is the centraliser of $g \in G$. Complementing the case of the universal calculus in Section 3, we focus next on the other extreme where $\mathcal{C}$ is a single conjugacy class stable under group inversion (such classes are called 'real' as the character is real-valued). Section 5 proves nondegeneracy for the natural 2-cycles calculus on symmetric groups S_n while Section 6 provides computer verification of this conjecture for real conjugacy classes up to $|G|$ some 75,000. This includes the smallest sporadic group M_{11} and the Suzuki group Sz_8. The geometric intuition is that when G is simple *any* $\mathcal{C}$ generates G and hence could be viewed as some kind of 'Lie algebra' which we might hope to reflect the simplicity of G.

Once we have a Killing form, nondegenerate or not, we can envisage many applications. In usual Lie theory complex simple Lie algebras have up to equivalence a unique compact real form where the Killing form is positive definite, leading in the geometry to a Riemannian metric with constant positive curvature. The simplest finite group examples along these lines have been studied 'by hand' and it was found, for example, that S_3 with its 2-cycles conjugacy class is an Einstein space with Ricci curvature essentially proportional to the metric, while A_4 with an order 4 conjugacy class is Ricci flat [11, 12]. With such eventual geometry in mind, we are interested in when the matrix of K in our basis $\mathcal{C}$ is positive definite. The answer among real conjugacy classes appears to be precisely those which are classes of involutions. One of these could then be a natural 'compact' braided Lie algebra structure of a finite nonabelian simple group. We find moreover that the other real conjugacy have the interesting feature of equal numbers of positive and negative eigenvalues when counted with multiplicity. We will also provide first results and conjectures on the integrality, irreducibility and maximal eigenvalue of the Killing form matrix, collected in Section 4.

We also explore some ideas relating to the open problem of associating an irrep to a conjugacy class. The underlying idea is that for a simple Lie algebra the adjoint representation is irreducible, so if a conjugacy class is like a 'Lie algebra' we might expect it to principally consist of an irrep when the group is simple or close to it (such as the symmetric group). We are thus motivated to speculate that for S_n and all simple groups where the Roth property holds one can bijectively associate irreps to conjugacy classes containing them. We illustrate this in Section 4 for A_5 and the Mathieu group M_{11}. In Section 5 we look in detail at S_n, where we rapidly establish that such a bijection, if it exists, is not any obvious one such as taking the Young

diagram associated to a conjugacy class and constructing a Specht module from the transposed diagram. However in Section 5.1 we find that the related question of which conjugacy classes contain the sign representation has a very natural answer.

We note that a first attempt at a geometrical association between conjugacy classes and irreps via the differential calculus was made in [10] and was equally tentative. An alternative strategy suggested by our positive-definiteness conjecture is to fix the noncommutative differential geometry as defined by such conjugacy class and to then view different conjugacy classes as leading to irreps in a less direct way as some kind of 'orbit method'.

Finally, Section 6 collects the computer data testing the ideas and conjectures in earlier sections on nonabelian finite simple groups up to order 75,000.

2. Preliminaries on noncommutative differential calculi

In this section we introduce the basic elements of noncommutative differential geometry of finite groups G in the Hopf algebraic approach. It is quite important that these constructions are not invented in an ad-hoc manner just for groups but that they are part of a general quantum groups approach to noncommutative differential geometry. Suffice it to say on that front that differential structures over any unital algebra A are expressed as a specification of the space Ω^1 of 1-forms as an $A-A$-bimodule equipped with a differential $\mathrm{d}: A \to \Omega^1$ obeying the Leibniz rule. We also require that the map $A \otimes A \to \Omega^1$ sending $a \otimes b \mapsto a\mathrm{d}b$ is surjective and, optionally (one says that the calculus is connected) that $\ker(\mathrm{d})$ is spanned by 1. When A is a Hopf algebra or 'quantum group' we can require the calculus to be covariant under left or right translation, or both. In the latter case one says that the calculus is bicovariant. One knows that a (say) left-covariant calculus is a free module over its space Λ^1 of left-invariant differentials. In the bicovariant case there is a canonical extension to a full exterior algebra (Ω, d) or 'noncommutative de Rahm complex'. Every unital algebra has a universal differential calculus defined as $\Omega^1 = \ker(\cdot : A \otimes A \to A)$ and $\mathrm{d}a = 1 \otimes a - a \otimes 1$. In the Hopf algebra case it is automatically bicovariant. Any other calculus is isomorphic to a quotient of the universal one.

Specialising these ideas, differential calculi on finite sets are in correspondence with directed graphs [1]. In the case of a finite group G the left-covariant calculi are in correspondence with Cayley graphs defined by non-empty $\mathcal{C} \subset G$ with $e \notin \mathcal{C}$ (e the group identity). Here the edges are of the form $x \to xa$ for $x \in G$ and $a \in \mathcal{C}$. Such a calculus is also right covariant (hence 'bicovariant') *iff* $\mathcal{C}$ is ad-stable. The basis of invariant 1-forms is $\{e_a \mid a \in \mathcal{C}\}$ and

$$\Omega = \mathbb{C}(G).\Lambda, \quad \Lambda = \oplus\Lambda^n, \quad \Lambda^0 = \mathbb{C}, \quad \Lambda^1 = \mathrm{span}\{e_a\},$$

$$\mathrm{d}f = \sum_a (\partial^a f)e_a, \quad e_a f = R_a(f)e_a$$

where $R_a(f) = f((\ )a)$ and $\partial^a = R_a - \mathrm{id}$ are right translation and 'left-invariant derivative' respectively. Explicitly, $e_a = \sum_x \delta_x \mathrm{d}\delta_{xa}$ in terms of Kronecker δ-functions at $x \in G$. The higher Λ are determined in the simplest 'Woronowicz' formulation by skew-symmetrization with respect to a braiding operator

$$\Psi(e_a \otimes e_b) = e_{aba^{-1}} \otimes e_a.$$

The calculus is connected i.e. has $H^0 = \mathbb{C}.1$ iff the graph is connected, which is iff $\mathcal{C}$ is a generating set. We refer to [8] for an introduction to the theory.

A nontrivial application of this theory to finite groups appeared in [10] where the exterior algebra was linked to the Fomin-Kirillov algebra arising in the study of the cohomology of flag varieties. The theme there is a kind of extension of Schur-Weyl duality to a duality between the geometry of the flag variety and that of the associated Weyl group.

Proposition 2.1. *A finite group G is simple* iff *all its bicovariant calculi are connected.*

Proof. Suppose that G is simple and $\mathcal{C}$ an ad-stable subset (defining a bicovariant calculus). Let $N = \langle\mathcal{C}\rangle$ the subgroup generated by $\mathcal{C}$. This is clearly normal and contains more than e (as $\mathcal{C}$ is nonempty), hence $N = G$ and the calculus is connected. Conversely, suppose that all nonempty ad-stable subsets $\mathcal{C}$ generate G. Let $N \subseteq G$ be normal and $\mathcal{C} = N \setminus \{e\}$. This is an ad-stable subset and $\langle\mathcal{C}\rangle = N$ as $N \neq \{e\}$ is a normal subgroup, hence $N = G$. □

Next, in the case of finite sets, a morphism $\phi : \Omega^1 \to \Omega^{1\prime}$ between two differential calculi (a bimodule map forming a commutative triangle with the corresponding d, d') is fully determined if it exists. This is because $\phi(\delta_x \mathrm{d}\delta_y) = \delta_x \mathrm{d}'\delta_y$ and hence sends an edge of the graph Γ of Ω^1 to the same edge in the graph Γ' of $\Omega^{1\prime}$ if this edge exists, or else to zero. Hence such a map is a surjection iff $\Gamma' \subseteq \Gamma$. Because of this, we see that $\mathcal{C}$ being a conjugacy class corresponds to Ω^1 having no proper quotients.

In order to develop a Lie theory, however, we need something slightly different. Namely, we impose connectedness from the start and in this case the 'minimal' differential calculus with no connected proper quotients is not necessarily given by a conjugacy class, for example, if the group is abelian. Dually, $\mathcal{C}$ could be required to be minimal but still generating.

Definition 2.2. A *minimal Lie structure* on a finite group G means a minimal ad-stable generating subset $\mathcal{C} \subset G$.

A minimal Lie structure always exists, namely one can start with $\mathcal{C} = G \setminus \{e\}$ (which defines the universal calculus), this is always ad-stable and clearly generates G. We then remove elements of $\mathcal{C}$ if possible until further removal is no longer possible. It does not contain e as this could be removed and hence $\mathcal{C}$ would not be minimal.

Proposition 2.3. *A finite group is simple* iff *every nontrivial conjugacy class is a minimal Lie structure. In this case the two notions are equivalent.*

Proof. Suppose that G is simple and $\mathcal{C}$ a nontrivial conjugacy class. As $\langle\mathcal{C}\rangle$ is normal and nontrivial we see, as above, that $\mathcal{C}$ generates G. Suppose that $\mathcal{C}$ is not minimal, so there exists a proper ad-stable generating subset $\mathcal{C}' \subset \mathcal{C}$ which, in particular, contradicts $\mathcal{C}$ a conjugacy class. Conversely, suppose that G is a finite group such that every nontrivial conjugacy class is a minimal Lie structure. Let $N \subset G$ be a nontrivial normal subgroup and $\mathcal{C} = N \setminus \{e\}$, which is therefore ad-stable. Decompose this into nontrivial conjugacy classes and let $\mathcal{C}'$ be one of

these. Then by assumption $\mathcal{C}'$ is a minimal Lie structure on G, hence $G = \langle \mathcal{C}' \rangle$. But $\langle \mathcal{C}' \rangle \subseteq N$ as N was a subgroup, hence $G \subseteq N$ and hence $N = G$. Hence G is simple. Finally, suppose that G is simple and $\mathcal{C}$ a minimal Lie structure. If $\mathcal{C}' \subset \mathcal{C}$ is a proper ad-stable subset then (as above) $N = \langle \mathcal{C}' \rangle$ is a nontrivial normal subgroup and hence $\mathcal{C}'$ generates G, which contradicts minimality. Hence $\mathcal{C}$ is a conjugacy class, nontrivial as does not contain e. $\square$

2.1. **Braided Lie algebras and Killing form.** In geometry the dual of the space Λ^1 of left-invariant differential 1-forms on a Lie group has the structure of a Lie algebra. The corresponding notion in the quantum groups approach to noncommutative differential geometry is that of a 'braided Lie algebra' [9]. This is defined abstractly as an object $\mathcal{L}$ in a braided monoidal category equipped with morphisms $[\ ,\ ] : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ (the 'Lie bracket') and $\Delta : \mathcal{L} \to \mathcal{L} \otimes \mathcal{L}$, $\epsilon : \mathcal{L} \to \underline{1}$ forming a coalgebra in the category (where $\underline{1}$ is the trivial object), subject to several axioms. When the category is abelian, there is a certain quadratic 'enveloping algebra' $U(\mathcal{L})$ associated to the braided Lie algebra and forming a bialgebra in the category. Of further relevance to us is a 'braided Killing form' defined as the braided or categorical trace of two application of $[\ ,\ ]$. The theory was invented particularly to solve the problem of what is the 'Lie object' underlying the standard Drinfeld-Jimbo quantum enveloping algebras $U_q(\mathfrak{g})$. It is shown in [9] that at least for generic q there is a braided Lie algebra $\mathcal{L} \subset U_q(\mathfrak{g})$ and an algebra surjection $U(\mathcal{L}) \to U_q(\mathfrak{g})$. In the classical limit $\mathcal{L} \subset U(\mathfrak{g})$ where $\mathcal{L} = \mathbb{C}1 \oplus \mathfrak{g}$ recovers the classical Lie algebra in an extended form where $[1,\ ] = \mathrm{id}$ and $[\ ,1] = 0$. An introduction to the theory is in [8]. This theory was connected to noncommutative differential geometry in [11, 6], where the general theorem is that for any 'inner' bicovariant differential calculus Ω^1 on a coquasitriangular Hopf algebra, there is a braided Lie algebra $\mathcal{L}$ isomorphic as an object in a braided category to Λ^{1*}. This is the case for all standard quantum group coordinate algebras $\mathbb{C}_q[G]$ and completes the picture above. The term 'inner' refers to a bi-invariant 1-form θ that generates the exterior derivative in the form $\mathrm{d} = [\theta,\ ]$, a concept that has no analogue in classical differential geometry. One could view the classical limit $\mathcal{L} = \mathbb{C}1 \oplus \mathfrak{g}$ as corresponding to a nonstandard higher order differential structure possible within noncommutative geometry.

Now let $\Omega^1(G)$ be a bicovariant differential calculus defined by an ad-stable subset $\mathcal{C}$ and with space of left-invariant 1-forms Λ^1. Let $\{x_a\}$ be dual basis to the classes of $\{e_a\}$ providing the basis of Λ^1. The bi-invariant element $\theta = \sum_{a \in \mathcal{C}} e_a$ makes the calculus inner and the associated 'braided Lie bracket' on $\mathcal{L} = \Lambda^{1*} = \mathbb{C}\mathcal{C}$ is $[x_a, x_b] = x_{aba^{-1}}$. The coproduct and counit take the form $\Delta x_a = x_a \otimes x_a$, $\epsilon(x_a) = 1$ and the category is that of G-modules with trivial braiding, the usual 'flip'. The associated 'enveloping algebra' $U(\mathcal{L})$ is a quadratic algebra with generators $\{x_a\}$ and relations $x_a x_b = x_{aba^{-1}} x_a$. This comes equipped with a canonical homomorphism $U(\mathcal{L}) \to \mathbb{C}G$ to the group algebra, defined by $x_a \mapsto a$. In our case because the underlying braiding is trivial the braided trace becomes the usual trace. Then the Killing form is

$$K(x_a, x_b) = \mathrm{Trace}_{\mathcal{L}}([x_a, [x_b,\ ]]) = |Z(ab) \cap \mathcal{C}| = \chi_{\mathcal{L}}(ab) \tag{2.1}$$

where $Z(g)$ is the centraliser of $g \in G$ and $\chi_{\mathcal{L}}$ is the character of the conjugation representation on $\mathcal{L}$. Clearly K is ad-invariant since $\mathcal{C}$ is and is Ψ-symmetric, hence

$\wedge(K) = 0$. It is also actually symmetric since $\chi_{\mathcal{L}}(ba) = \chi_{\mathcal{L}}(a(ba)a^{-1}) = \chi_{\mathcal{L}}(ab)$ since $\chi_{\mathcal{L}}$ is a class function. We are interested in when K is nondegenerate.

Lemma 2.4. *If $\mathcal{C} \cap (\mathcal{C}.c) \neq \emptyset$ for some nontrivial $c \in Z(G)$ then K is degenerate. In particular, if $|Z(G) \cap \mathcal{C}| > 1$ then K is degenerate.*

Proof. Looking at K as a matrix with rows and columns labelled by $\mathcal{C}$. If $b = b'c$ where $b, b' \in \mathcal{C}$ and $c \in Z(G) \setminus \{e\}$ then $K(x_a, x_b) = |Z(ab) \cap \mathcal{C}| = |Z(abc) \cap \mathcal{C}| = K(x_a, x_{b'})$ for all $a \in \mathcal{C}$, hence K has a repeated column. If $b, b' \in Z(G) \cap \mathcal{C}$ are distinct then $c = b'^{-1}b$ fits the first part. □

Corollary 2.5. *If $|G| > 2$ and $|Z(G)| > 1$ then the Killing form for the universal calculus is degenerate.*

Proof. For the universal calculus $\mathcal{C} = G \setminus \{e\}$ and in the preceding lemma we can take any nontrivial $c \in Z(G)$, any $b' \neq c^{-1}, e$ and $b = b'c$. Then $b \in \mathcal{C} \cap (\mathcal{C}c)$. □

If G has order 2 then K *is* a 1×1 matrix and *is* nondegenerate. We therefore only need to investigate the universal calculus in the case where $|G| > 2$ and $Z(G) = \{e\}$, for example when G is simple and nonabelian.

3. Nondegeneracy for the universal calculus when Roth's property holds

Let G be a finite group. We are going to work from the expression (2.1) derived above, and we note first that this formulation actually makes sense for a 'Killing form' similarly defined for any representation W and on all of $\mathbb{C}G$,

$$K_W(x_a, x_b) := \chi_W(ab)$$

where $\{x_a \mid a \in G\}$ is a basis of $\mathbb{C}G$. It is well-known that such a symmetric bilinear form is nondegenerate if and only if W contains every irrep of G with positive multiplicity. This follows from semisimplicity of the group algebra $\mathbb{C}G$ and general facts about semisimple algebras. (Namely, if an algebra is semisimple then it is a direct sum of matrix blocks. If $W = \oplus_i n_i V_i$ for some multiplicities n_i then K_W has a block form with n_i times the Euclidean inner product on each matrix block. This is because an element in a matrix block corresponding to an irrep acts as zero by left multiplication on any other block and hence in any other irrep. Hence K_W is nondegenerate iff all the $n_i > 0$). The case $W = \mathbb{C}G$ with the conjugation representation is therefore nondegenerate *iff* the conjugation representation contains every irrep. If $Z(G)$ is trivial then this is the Roth property. For the case of the universal calculus we are interested in the case $W = \mathbb{C}.(G\setminus\{e\})$ where we remove the group identity. We are also interested in restricting the Killing form to $\mathbb{C}.(G \setminus \{e\})$ but we defer this for the moment.

Lemma 3.1. *Let G be a finite group. Suppose $\mathbb{C}G$ under conjugation contains every irreducible of G with strictly positive multiplicity. Then so does $W = \mathbb{C}.(G \setminus \{e\})$.*

Proof. W spanned by set $G \setminus \{e\}$ still contains a copy of the trivial representation since this set is permuted by conjugation and hence the element $\theta = \sum_a x_a$, where we sum over $a \in G \setminus \{e\}$, is invariant. As $\mathbb{C}G = W \oplus \mathbb{C}.e$ as a G-module, if any nontrivial representation is contained in $\mathbb{C}G$ then it must also be in W. □

Hence if the Roth property holds for G then it also holds for our particular W of interest. Let n be the number of distinct irreducible representations of G, and call these representations $V_1, \dots, V_n$.

Proposition 3.2. *Let G be a finite group and W be a representation of G for which every irreducible representation V_i appears in W with strictly positive multiplicity. Then the restriction of K_W to $\mathbb{C}.(G \setminus \{e\})$ is nondegenerate.*

Proof. Since the form K_W is nondegenerate on $\mathbb{C}G$ we know that $(\mathbb{C}.(G \setminus \{e\}))^\perp$ is one-dimensional. We will know that K_W is nondegenerate in $\mathbb{C}.(G \setminus \{e\})$ if there is no element in the perpendicular which also lies $\mathbb{C}.(G \setminus \{e\})$. We prove this by determining explicitly a vector spanning the line $(\mathbb{C}.(G \setminus \{e\}))^\perp$, and observing that it doesn't lie in $\mathbb{C}.(G \setminus \{e\})$.

Define

$$m = \sum_{g \in G} \left(\sum_{i=1}^{n} \frac{\dim(V_i)^2}{\langle \chi_{V_i}, \chi_W \rangle} \overline{\chi_{V_i}(g)} \right) x_g$$

This is well-defined since $\langle \chi_{V_i}, \chi_W \rangle \neq 0$ for all i by our assumption.

We claim that $K_W(x_a, m) = 0$ for all $a \neq e$. Note that the coefficient of x_e in m is given by the formula

$$m_e = \sum_{i=1}^{n} \frac{\dim(V_i)^3}{\langle \chi_{V_i}, \chi_W \rangle},$$

so is always strictly positive. Therefore m does not lie in $\mathbb{C}.(G \setminus \{e\})$ and the claim will imply the proposition.

We use the following standard orthogonality formulas:

$$\sum_{g \in G} \overline{\chi_V(g)}\, \chi_{V'}(ga) = \begin{cases} 0 & \text{if } V, V' \text{ are distinct irreps,} \\ \frac{|G|}{\dim V} \chi_V(a) & \text{otherwise,} \end{cases} \tag{3.1}$$

and

$$\sum_i \overline{\chi_{V_i}(a_1)} \chi_{V_i}(a_2) = 0 \quad \text{if } a_1 \text{ and } a_2 \text{ are not conjugate.} \tag{3.2}$$

Note also that

$$\chi_W(g) = \sum_i \langle \chi_{V_i}, \chi_W \rangle\, \chi_{V_i}(g).$$

Now we can compute, extending linearly,

$$
\begin{aligned}
K_W(m, x_a) &= K_W\left(\sum_{g\in G}\left(\sum_{i=1}^{n}\frac{\dim(V_i)^2}{\langle\chi_{V_i},\chi_W\rangle}\overline{\chi_{V_i}(g)}\right)x_g, x_a\right)\\
&= \sum_{g\in G}\left(\sum_{i=1}^{n}\frac{\dim(V_i)^2}{\langle\chi_{V_i},\chi_W\rangle}\overline{\chi_{V_i}(g)}\right)\chi_W(ga) = \sum_{i=1}^{n}\frac{\dim(V_i)^2}{\langle\chi_{V_i},\chi_W\rangle}\left(\sum_{g\in G}\overline{\chi_{V_i}(g)}\chi_W(ga)\right)\\
&= \sum_{i=1}^{n}\dim(V_i)^2\left(\sum_{g\in G}\overline{\chi_{V_i}(g)}\chi_{V_i}(ga)\right) = |G|\sum_i \dim(V_i)\chi_{V_i}(a)\\
&= |G|\sum_i \overline{\chi_{V_i}(e)}\chi_{V_i}(a).
\end{aligned}
$$

The last expression vanishes whenever $a \neq e$ by (3.2). □

Corollary 3.3. *Let G be a finite group for which the Roth property holds. Then the Killing form for the universal differential calculus on $\mathbb{C}(G)$ is nondegenerate.*

As explained in the introduction, the Roth property holds for most simple non-abelian groups including all the sporadic ones, see [7]. Meanwhile, the simplest example where the Roth property holds is $G = S_3$, the group of permutations on three elements. This is elementary enough that we can, instructively, compute everything in our characters approach by hand. Here $Z(G)$ is trivial.

Example 3.4. Let $G = S_3$. The Killing form on $\mathbb{C}G$ is

$$
K_W = \begin{pmatrix} 5&1&1&1&2&2\\ 1&5&2&2&1&1\\ 1&2&5&2&1&1\\ 1&2&2&5&1&1\\ 2&1&1&1&2&5\\ 2&1&1&1&5&2 \end{pmatrix}
$$

in a basis $e, u = (12), v = (23), w = (13) = uvu, uv = (123)$ and $vu = (132)$. We obtained by writing out the group product table and evaluated $\chi_W(g) = |Z(g)| - 1$ for each entry $g \in S_3$. Here

$$\chi_W(e) = 5, \quad \chi_W(u) = \chi_W(v) = \chi_W(w) = 1, \quad \chi_W(uv) = \chi_W(vu) = 2.$$

One can then see by computation that the lower right 5×5 block in K_W is invertible as required by the Proposition. To see how this arises from the character theory in the proposition, we look at the trivial representation V_{triv}, the sign (parity function) representation V_{sgn} and the standard 2-dimensional representation, denoted V_Δ. Their character values tabulated on conjugacy classes as for χ_W are

$$
\begin{aligned}
\chi_{triv} &= 1, 1, 1\\
\chi_{sgn} &= 1, -1, 1\\
\chi_\Delta &= 2, 0, 1
\end{aligned}
$$

from which we see that $W = 2V_{triv} \oplus V_{sgn} \oplus V_\Delta$. Thus the 'Roth property' holds for W as in the Lemma above, hence K_W on $\mathbb{C}G$ is nondegenerate. We also view this last table as the change of basis with the characteristic class functions θ_0 with

support $\{e\}$, θ_1 with support $\{u,v,w\}$ and θ_2 with support $\{uv,vu\}$. Now, by direct computation it is clear that $\tilde{m}=19\theta_0-\theta_1-5\theta_2$ viewed as the coefficients of an element of $\mathbb{C}G$ lies in $(\mathbb{C}.(G\setminus\{e\}))^\perp$ and hence spans this 1-dimensional space. Using the table to convert back to the basis of characters, we have obtained 'by hand' an element $\tilde{m}=\chi_{triv}+2\chi_{sgn}+8\chi_\Delta$ in the perpendicular complement. This agrees up to normalisation with the canonical element constructed in the proof of the proposition, namely

$$m=\frac{1}{2}\chi_{triv}+\chi_{sgn}+4\chi_\Delta$$

where the coefficient of χ_V is $\dim(V)^2/\langle\chi_V,\chi_W\rangle$. Finally, $\tilde{m}$ has by construction a non-zero coefficient of e, so $\tilde{m}$ does not lie in $\mathbb{C}.(G\setminus\{e\})$.

In fact we conjecture that the conclusion of Proposition 3.2 holds even when the Roth property fails.

Conjecture 3.5. Let G be a finite simple nonabelian group and K the Killing form for the universal differential calculus. Then K is nondegenerate.

Experimental evidence that this holds for the finite simple groups beyond those implied by Proposition 3.2 is limited as even the smallest of the simple groups such as $PSU(3,4)$ where the Roth property is known to fail in [7] require too much computer memory to check directly. However, we will have evidence for a more general conjecture where the Roth property for the relevant representation and hence Proposition 3.2 do not apply but where the Killing form is still nondegenerate. In particular, we will see in Section 6 that nondegeneracy holds for all conjugacy class differential calculi on $PSU(3,4)$. Also, just as the Roth property holds more generally than for certain simple groups, eg for S_3 above, we similarly further conjecture that the nondegeneracy may hold more generally for finite groups with trivial centre.

Finally, we also conjecture nondegeneracy at the other extreme of 'minimal' differential calculi. Note that for a simple group any conjugacy class will be generating and define a connected differential calculus.

Conjecture 3.6. Let G be a finite simple nonabelian group and $\mathcal{C}\subseteq G\setminus\{e\}$ a conjugacy class closed under inversion. Then the Killing form K on $\mathbb{C}\mathcal{C}$ is nondegenerate.

This is verified by computer in Section 5 for groups up to order 75,000. This includes the first sporadic group M_{11} and the exceptional group of Lie type Sz_8. The requirement on having inverses is included since the experimental evidence shows examples of conjugacy classes without this assumption and where K is degenerate, although those examples are scarce and generally one has nondegeneracy even without requiring closure under inversion.

Conjecture 3.6 and our previous Conjecture 3.5 suggest further that nondegeneracy holds for simple G and any ad-stable subset $\mathcal{C}\subseteq G\setminus\{e\}$ closed under inverses, but we have no direct evidence for this. We also further conjecture that nondegeneracy can hold more generally for groups which may have centre but where $Z(G)\cap\mathcal{C}$ is empty and $\mathcal{C}$ generates the group. For example, S_n with its 2-cycles conjugacy class

(which generates the group) has nondegenerate Killing form yet the analogue of the 'Roth property' clearly does not hold for $W = \mathbb{C}\mathcal{C}$ as most irreps are not contained in it. We will prove these assertions in Section 5, but for now we directly look at the simplest case of S_3 to illustrate why we require $\mathcal{C}$ to generate the group.

Example 3.7. For $G = S_3$ and $\mathcal{C}$ the order 3 conjugacy class of 2-cycles (which generates the group), the Killing form is nondegenerate. Here G acting by conjugation permutes the elements of $\mathcal{C} = \{u, v, w\}$ and hence $W = V_{triv} \oplus V_\Delta$. This does not have the Roth-type property used in Proposition 3.2 and indeed on $\mathbb{C}G$ we have $\chi_W = 3, 1, 2$ on the three classes and hence

$$K_W = \begin{pmatrix} 3&1&1&1&0&0\\ 1&3&0&0&1&1\\ 1&0&3&0&1&1\\ 1&0&0&3&1&1\\ 0&1&1&1&0&3\\ 0&1&1&1&3&0 \end{pmatrix}$$

in basis order e, u, v, w, uv, vu as before, which is degenerate. However, the 3×3 submatrix arising from its restriction to $\mathcal{C}$ is a multiple of the Euclidean inner product [10] and is invertible. Incidentally, its restriction to the other class and to the union of the two nontrivial classes is also nondegenerate.

Example 3.8. For $G = S_3$ and $\mathcal{C}$ the order 2 conjugacy class of 3-cycles (which does not generate the group), the Killing form is degenerate. Here each element u, v, w acts to flip the elements of $\mathcal{C}$ while uv, vu, e of course act trivially. Hence $W = V_{triv} \oplus V_{sgn}$ again does not have the Roth-type property. In this case $\chi_W = 2, 0, 2$ on the three classes. Hence

$$K_W = \begin{pmatrix} 2&0&0&0&2&2\\ 0&2&2&2&0&0\\ 0&2&2&2&0&0\\ 0&2&2&2&0&0\\ 2&0&0&0&2&2\\ 2&0&0&0&2&2 \end{pmatrix}$$

which is degenerate. Its 2×2 submatrix corresponding to the restriction to $\mathcal{C}$ is also degenerate (as is its restriction to the other nontrivial conjugacy class and to the sum of the two classes).

4. Eigenvalues of the Killing form operator associated to a conjugacy class

We now look at some general results about the Killing form of particular relevance to the case of a conjugacy class differential calculus. Let G be a a finite group and $\mathcal{C}$ an ad-stable subset of $G \setminus \{e\}$. We note that the 'Euclidean metric' $\eta(x_a, x_b) = \delta_{a,b}$ given by the Kronecker delta-function in basis $\mathcal{C}$ is also ad-invariant. We regard both η, K as maps $\mathbb{C}\mathcal{C} \to (\mathbb{C}\mathcal{C})^*$ and hence the composite $\eta^{-1} \circ K$, which we also denote K, is a map $\mathbb{C}\mathcal{C} \to \mathbb{C}\mathcal{C}$. Here explicitly

$$K(x_a) = \sum_{b \in \mathcal{C}} K(x_a, x_b) x_b.$$

By construction this is also ad-invariant. Hence its eigenspaces provide a natural decomposition of $\mathbb{C}\mathcal{C}$ into subrepresentations. Note that as K is real and symmetric in our basis it has a full diagonalisation with real eigenvalues over $\mathbb{R}$. However, it can also be viewed as a hermitian matrix or self-adjoint operator over $\mathbb{C}$. Also the entries of K are non-negative integers.

Proposition 4.1. *Suppose V is an irreducible representation of $\mathbb{C}G$ which is defined over $\mathbb{Q}$. So $V = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ for an irreducible representation $V_{\mathbb{Q}}$ of $\mathbb{Q}G$. Furthermore, suppose $\mathcal{C}$ is a conjugacy class in G such that V occurs in the conjugation representation $\mathbb{C}\mathcal{C}$.*

If the isotypical component of V in $\mathbb{C}\mathcal{C}$ is contained in a single eigenspace of the Killing matrix K of $\mathcal{C}$, then the corresponding eigenvalue lies in $\mathbb{Z}$.

Proof. Choose an element $x \in \mathcal{C}$ and consider the map

$$\pi : \mathbb{C}G \to \mathbb{C}\mathcal{C} : g \mapsto gxg^{-1}$$

which is a G-equivariant surjection from the left-regular representation to the conjugation representation of G. Let $A_V \subset \mathbb{C}G$ denote the block of the irreducible representation V. By block decomposition of $\mathbb{C}G$ and Schur's lemma it follows that π restricts to a surjection from A_V to the isotypical component of V in $\mathbb{C}\mathcal{C}$. However, since V was defined over $\mathbb{Q}$ it follows that A_V has a basis that lies inside $\mathbb{Q}G$. Moreover by surjectivity of $\pi|_{A_V}$ onto the isotypical component there exists such a basis element b whose image $\pi(b)$ is nonzero. By the assumptions, b is an eigenvector of the Killing matrix. Moreover b has rational coefficients as a vector in $\mathbb{C}\mathcal{C}$. Since the entries of K are integral and b is rational it follows that the eigenvalue of b lies in $\mathbb{Q}$. On the other hand the integrality of K implies that the eigenvalues are all algebraic integers. So the eigenvalue of b is a rational number and an algebraic integer. Therefore it must lie in $\mathbb{Z}$. $\square$

Note that this proposition implies in particular, that if V is a complex representation of G defined over $\mathbb{Q}$ which occurs in $\mathbb{C}\mathcal{C}$ with multiplicity 1, then it lies in an eigenspace of K with eigenvalue in $\mathbb{Z}$. Since all representations of the symmetric group are defined over $\mathbb{Q}$ (over $\mathbb{Z}$ even), we have the following corollary.

Corollary 4.2. *Let $\mathcal{C}$ be a nontrivial conjugacy class of S_n. If an irreducible representation of S_n occurs in the conjugation representation $\mathbb{C}\mathcal{C}$ with multiplicity one, then it embeds into an eigenspace for the corresponding Killing form with eigenvalue in $\mathbb{Z}$.* $\square$

Note that an irreducible representation is rational if all its character values are rational. This is because the matrix entries can be obtained by projection via central idempotents in the group algebra with coefficients defined by the characters. Similarly the character determines if an irreducible representation is complex in the sense of not real.

Proposition 4.3. *Let $\mathcal{C} \subseteq G \setminus \{e\}$ be an ad-stable subset.*

1. *If a complex irreducible representation V occurs in $\mathbb{C}\mathcal{C}$ in an eigenspace of the associated Killing form matrix, then so does its dual representation (with complex conjugate character).*

(2) *If we consider the inverse conjugacy class $\mathcal{C}^{-1}$ then the eigenvalues of the Killing form matrix for $\mathcal{C}^{-1}$ are the same as the ones obtained for $\mathcal{C}$, and the decompositions of the respective eigenspaces into irreps are isomorphic.*

Proof. Let $\mathcal{C}$ be an ad-stable subset. The conjugation representation is clearly defined over $\mathbb{R}$ by working in the basis $\mathcal{C}$ (indeed, over $\mathbb{Z}$). Since K is real and symmetric in the basis $\mathcal{C}$ its eigenspaces are also defined over $\mathbb{R}$, and hence real as subrepresentations of the conjugation representations. This implies the first part. For the second part we consider inversion as a bijection between the two ad-stable subsets. Let $a, b, c \in \mathcal{C}$. Clearly c commutes with ab *iff* c^{-1} commutes with $b^{-1}a^{-1}$. But as the Killing forms are symmetric, we see that the Killing forms have the same matrices in their respective bases. If $v \in \mathbb{C}\mathcal{C}$ is expanded in the basis $\mathcal{C}$ we define $\tilde{v}$ to be the corresponding vector in $\mathbb{C}\mathcal{C}^{-1}$ with the same coefficients in the corresponding basis, i.e. $v, \tilde{v}$ are represented by the same column vector in their respective bases. One may readily see that the matrices for the action of an element of g in the two cases are also identical. This implies the second part. $\square$

One can extend the first part to a general Galois theory argument that if an irrational representation occurs then so do its various 'conjugates', with the corresponding change of Galois root in the eigenvalue (we will see this for A_5 below).

Next, we look at the trivial representation in $\mathbb{C}\mathcal{C}$ canonically present as spanned by the sum of the elements of $\mathcal{C}$. By a slight abuse of notation we denote this element by θ (its analogue as a left-invariant 1-form makes the calculus inner). We recall that a matrix with non-negative entries is called *irreducible* if for all indices i, j there exists $m \in \mathbb{N}$ such that the matrix entry $(K^m)_{ij} \neq 0$. This is equivalent to connectedness of the graph on the set of indices defined by an edge whenever the entry $K_{ij} \neq 0$.

Proposition 4.4. *Let G be a finite group and $\mathcal{C} \subseteq G \setminus \{e\}$ a conjugacy class. Then K has a (positive) integral maximal eigenvalue c, given by the sum of any row of K. Moreover, K splits onto r irreducible components* iff *the eigenspace associated to c has dimension r and in this case all other eigenspace dimensions are divisible by r. In particular, if K is irreducible then the eigenspace associated to c is 1-dimensional, generated by the eigenvector $\theta = \sum_a x_a$.*

Proof. $K(\theta) = \sum_{a,b} K(x_a, x_b)x_b = \sum_b c_b x_b$ where c_b is the sum of the b'th column of the matrix of K. However, $c_{gbg^{-1}} = \sum_a K(x_a, x_{gbg^{-1}}) = \sum_a K(x_{g^{-1}ag}, x_b) = c_b$ after a change of variables. Hence c_b is independent of $b \in \mathcal{C}$ in the case of a conjugacy class. Hence θ is an eigenvector of K with eigenvalue as stated. Moreover, if K is irreducible then by Perron-Frobenius theory there is a 1-dimensional maximal eigenspace with eigenvalue the column sum of K, i.e. with eigenvector θ. If K is not irreducible then after a reordering of the basis it can be presented as a direct sum. Iterating this, we reduce K to a direct sum of some number $r > 1$ of irreducible blocks. In fact each block will be, after reordering, a copy of the same irreducible matrix. This follows from ad-invariance of K as follows. Consider an element in G that conjugates a corner of the first block to the corresponding corner of another. All the indices relating to the first block belong to the same connected component of the graph and, by assumption, they are not connected to any of the indices

for the other blocks, and this notion is ad-invariant, as K is. Hence the indices relating to the conjugated first block must be connected to themselves and not to the first block. Hence the first block maps over to the conjugated block, and all its entries are the same when suitably ordered, again by ad-invariance of K. Once K has been presented as r blocks K_i, its eigenvectors will consist of r parts forming eigenvectors for each block with the same eigenvalue. However, since these blocks are all irreducible and have the same row sum as K, they will each have the same maximal eigenvalue as K, and any other eigenvalues will be strictly lower. This implies the facts stated. □

Note that the diagonal of K is always non-zero as a commutes with a^2 for all $a \in \mathcal{C}$. Hence K^{m+1} can only have the same or more positive entries as K^m, so in our case irreducible is equivalent to the existence of $m \in \mathbb{N}$ such that all entries of K^m are positive, i.e. the *primitivity* of the matrix K.

Conjecture 4.5. Let G be a finite nonabelian simple group and $\mathcal{C} \subseteq G \setminus \{e\}$ a conjugacy class not consisting of involutions. Then the associated K is irreducible.

This is surmised by looking at finite simple groups up to order 75,000. The only observed reducible cases are the classes of involutions for $G = PSL(2, 2^k)$, $G = PSU(3, 2^k)$ or $G = Suz(2^{2k-1})$ for $k \geq 2$ up to the order that we could check. These are all groups of Lie type over finite fields of characteristic 2 and we further conjecture that this is always the case. Results for S_n will be discussed in Section 5.

Next, we consider the question: for which conjugacy classes is K not only nondegenerate but positive definite? We have explained in the introduction that if $\mathbb{C}\mathcal{C}$ is some kind of braided Lie algebra we might expect a simple group to have a unique 'compact real form', i.e. a choice of $\mathbb{C}\mathcal{C}$ with K positive definite. Whilst this is not generally true, we can conjecture a close enough result:

Conjecture 4.6. Let G be a finite nonabelian simple group and $\mathcal{C} \subseteq G \setminus \{e\}$ a conjugacy class closed under inverses. If $\mathcal{C}$ consists of involutions, then the Killing form is positive definite. Otherwise the Killing form has zero signature (i.e. the same number of positive and negative eigenvalues counted with multiplicities).

This is supported by computer verification to order 75,000. Note that by the Feit-Thompson theorem every (nonabelian) finite simple group must contain involutions, so a conjugacy class in the terms of the conjecture will always exist. Uniqueness is not true in general, since various simple groups (eg. the altenating groups A_n for $n \geq 8$) contain several different conjugacy classes of involutions. There are also some examples, see the conjugacy classes 3A and 3B in $PSU(4, 2)$ in Section 6, that yield positive definite Killing forms without being closed under inverses. We also expect a behaviour similar to the conjectured one to hold for groups such as S_n with $\mathcal{C}$ generating, see the next section.

The second part of Conjecture 4.6 is not supported by the analogy but it perhaps reminiscent of the split real form in Lie theory. Finally, bearing in mind that a braided-Lie algebra in the case of a classical Lie group has the form $\mathcal{L} = \mathbb{C}1 \oplus \mathfrak{g} \subset U(\mathfrak{g})$ where $\mathfrak{g}$ is in the adjoint representation and is irreducible when the Lie group is simple, we might hope that $\mathbb{C}\mathcal{C}$ will decompose in a similar way and that this might produce an irreducible representation analogous to $\mathfrak{g}$ after removal of the

$\mathcal{C} \subset A_4$	$\lvert\mathcal{C}\rvert$	Decomposition (eigenvalue)	Suggested irrep
(12)(34)	3	$1(9) \oplus 1^*(0) \oplus \bar{1}^*(0)$	$1^*, \bar{1}^*$
(123)	4	$1(4) \oplus 3(0)$	3
(124)	4	$1(4) \oplus 3(0)$	3

$\mathcal{C} \subset A_5$	$\lvert\mathcal{C}\rvert$	Decomposition (eigenvalue)	Suggested irrep
(12)(34)	15	$1(21) \oplus 4(21) \oplus 5(12) \oplus 5(12)$	4
(123)	20	$1(34) \oplus 4(24) \oplus 5(18) \oplus 4(-12) \oplus 3(-22) \oplus \bar{3}(-22)$	5
(12345)	12	$1(24) \oplus 5(12) \oplus 3(-10+2\sqrt{5}) \oplus \bar{3}(-10-2\sqrt{5})$	$3, \bar{3}$
(12354)	12	$1(24) \oplus 5(12) \oplus \bar{3}(-10+2\sqrt{5}) \oplus 3(-10-2\sqrt{5})$	$\bar{3}, 3$

Table 1. Decomposition of span of conjugacy classes into irreps (eigenvalue of Killing form in brackets), for A_4, A_5. Here irreps are labelled by their dimensions and other symbols, with a bar for the conjugate representation.

trivial representation. This is not exactly what happens but we propose to use the Killing form to pick out an irreducible component in $\mathbb{C}\mathcal{C}$, and we shall explore this for S_n with its 2-cycles class in the next section. By thinking of any conjugacy class as a braided-Lie algebra and using the Killing form we might hope to associate irreps to conjugacy classes in a natural way. This is not so much a matter of precise conjectures at the moment but a programme in the form of a couple of 'principles' which we propose.

Principle 4.7. (Nondegeneracy principle.) For large finite nonabelian simple groups G the decomposition into eigenspaces of K is typically into irreps or into complex conjugate pairs of irreps.

This is certainly not always true but we shall see how it works for the M_{11} group. Note that the strong form stated implies that where an irrep occurs with multiplicity then these copies are generally distinct as well, i.e. K typically resolves or 'separates' the isotypical components in the process.

Principle 4.8. (Correspondence principle.) Let G be a finite nonabelian simple group for which Roth's property holds. There is often a 'reasonable' way up to choice of Galois and complex conjugates to bijectively assign nontrivial irreps to non-trivial conjugacy classes containing them in the conjugation representation, in such a way that if a complex irrep is assigned to a conjugacy class not stable under inversion then its complex conjugate is assigned to the inverse class, and where possible the assigned irrep occurs without multiplicity.

We certainly find examples where avoiding multiplicities is not possible, and in this case a weaker requirement is to have a distinct eigenvalue of K for the chosen irrep or complex conjugate pair of irreps.

Example 4.9. Table 1 provides a first impression by looking at A_5. We used Sage and some deductive reasoning to determine the decomposition of conjugacy classes into irreps and their eigenvalues. There are some accidental degenerations, one of them because K is reducible, but otherwise the eigenspaces of K are irreps or sums of 'conjugate' irreps (though not in this case complex conjugate). And

in the last column we have also selected an irrep to associate to each conjugacy class in such a way that there is a 1-1 correspondence. This is a more or less unique assignment if we want the 5 and the 4 to enter in their respective conjugacy classes without multiplicity and *if* we intuitively require the 'conjugate' pair of 5-cycle classes to map to the 'Galois conjugate' pair of irreps (which of the pair of classes is assigned to which of $3, \bar{3}$ is a residual ambiguity). The eigenvalues have been deduced from the knowledge (see above) that the trivial representation has maximal eigenvalue followed by comparison of dimensions between eigenspaces and irreps. This method does not fully identify the eigenvalue when distinct irreps with the same dimension occur, but in the case of $3, \bar{3}$ this is a matter of definition as to which is called which. However, having defined one as 3 one may verify by explicit calculation that it appears in the last line of the table with the other eigenvalue. The eigenvalues in these classes are not integer and indeed the eigenspaces are examples of real irrational representations (however, when they both appear with the same eigenvalue then this is an integer). Also note that there is a unique conjugacy class where K is positive definite and the equal dimensions of the combined positive and negative eigenspaces in each of the other cases. We contrast both features with A_4, also in the table, where aside from the trivial representations all eigenvalues are zero. Note that A_4 is *not* simple and indeed there is no way to associate nontrivial irreps and nontrivial conjugacy classes containing them.

Example 4.10. Table 2 illustrates that the same ideas hold in the simplest sporadic group, M_{11}, again using Sage and deductive reasoning to construct the table. Here again we find for the most part that the eigenspace decomposition is generally a decomposition into irreps or conjugate pairs of irreps (the sole exception is the 10,11 irreps in the class $2A$). We use ATLAS labelling of conjugacy classes as in Section 6. We also find a more or less unique 1-1 correspondence between conjugacy classes and irreps as shown, as follows. We see as explained above that inverse pairs of conjugacy classes have the same spectrum and for any complex conjugate pair of irreps if one occurs in a conjugacy then so must the other and with the same eigenvalue. In view of this, inverse pairs of classes are naturally assigned to a pair of complex conjugate irreps as shown (which of each pair goes to which is then largely a matter of definition). This forces us to assign $10^*, 1\bar{0}^*$ as shown as it does not occur in the other inverse pair, and hence $16, \overline{16}$ to that. Both then occur without multiplicity. Of the remaining nontrivial irreps the only one which occurs without multiplicity in the 2nd row of the table is 11, so we are led to assign that. Similarly in the class $3A$, only the 10 occurs without multiplicity so we assign that. Of the remaining classes only the 55 appears in the class $2A$ without multiplicity, so we assign that. However, the assignment of the 44 and 45 is not determined by these arguments and this represents the residual ambiguity. They both occur with multiplicity in the remaining two conjugacy classes. One of these, the $5A$, has no nontrivial irreps occurring with integer eigenvalues at all, so this provides no guide either. Note also the unique conjugacy class $2A$ that has positive definite K and the equal dimensions of the combined positive and negative eisgenspaces in each of the other classes, other than the mutually inverse pair $8A, 8B$ and the mutually inverse pair $11A, 11B$.

Table 2 also shows the need for a more compact account of the Killing form eigenvalues and we do this in Figure 1 in the form of a spectrogram. This shows all the

$\mathcal{C} \subset M_{11}$	$\lvert\mathcal{C}\rvert$	Decomposition (eigenvalue)	Suggested irrep
$2A$	165	$1(489) \oplus 44(223.) \oplus 10(192) \oplus 11(192) \oplus 55(136) \oplus 44(122.)$	55
$3A$	440	$1(946) \oplus 11(609.) \oplus 10(572) \oplus 44(484.) \oplus 44(470.)$ $\oplus 55(428) \oplus 11(404.) \oplus 44(344.) \oplus 45(-413.) \oplus 10^*(-420)$ $\oplus \overline{10}^*(-420) \oplus 55(-426) \oplus 55(-444) \oplus 45(-448.)$	10
$4A$	990	$1(2108) \oplus 10(1195.) \oplus 44(1047.) \oplus 45(1043.) \oplus 44(1013.)$ $\oplus 11(1010) \oplus 55(996.) \oplus 44(988.) \oplus 16(986) \oplus \overline{16}(986)$ $\oplus 45(982.) \oplus 55(959.) \oplus 55(952) \oplus 44(949.) \oplus 10(918.)$ $\oplus 44(-910.) \oplus 55(-933.) \oplus 55(940.) \oplus 45(-977.) \oplus 55(-984.)$ $\oplus 44(-987.) \oplus 45(-1004.) \oplus 16(-1014) \oplus \overline{16}(-1014)$ $\oplus 55(-1025.) \oplus 10^*(-1026) \oplus \overline{10}^*(-1026) \oplus 45(-1062.)$	11
$5A$	1584	$1(3096) \oplus 11(1828.) \oplus 45(1712.) \oplus 55(1686.) \oplus 44(1685.)$ $\oplus 10(1673.) \oplus 11(1649.) \oplus 44(1628.) \oplus 55(1588.) \oplus 45(1586.)$ $\oplus 16(1575.) \oplus \overline{16}(1575.) \oplus 55(1574.) \oplus 44(1570.) \oplus 44(1556.)$ $\oplus 44(1544.) \oplus 11(1537.) \oplus 55(1534.) \oplus 16(1532.) \oplus \overline{16}(1532.)$ $\oplus 10(1526.) \oplus 55(1515.) \oplus 44(1506.) \oplus 45(1496.) \oplus 45(-1493.)$ $\oplus 55(-1508.) \oplus 10^*(-1527.) \oplus \overline{10}^*(-1527.) \oplus 55(-1534.)$ $\oplus 16(-1542.) \oplus \overline{16}(-1542.) \oplus 45(-1548.) \oplus 44(-1549.)$ $\oplus 55(-1562.) \oplus 45(-1567.) \oplus 55(-1572.) \oplus 16(-1573.)$ $\oplus \overline{16}(-1573.) \oplus 45(-1574.) \oplus 10^*(-1576.) \oplus \overline{10}^*(-1576.)$ $\oplus 45(-1591.) \oplus 55(-1617.) \oplus 44(-1622.) \oplus 45(-1638.)$ $\oplus 55(-1775.)$	$44, 45$
$6A$	1380	$1(2568) \oplus 44(1532.) \oplus 11(1465.) \oplus 44(1380.) \oplus 10(1366.)$ $\oplus 44(1359.) \oplus 55(1335.) \oplus 44(1324.) \oplus 16(1320) \oplus \overline{16}(1320)$ $\oplus 11(1315.) \oplus 10(1313.) \oplus 55(1305.) \oplus 44(1298.) \oplus 45(1298)$ $\oplus 55(1284) \oplus 55(1276) \oplus 55(1276.) \oplus 45(1256) \oplus 44(1251.)$ $\oplus 11(1226.) \oplus 55(-1173.) \oplus 45(-1214.) \oplus 55(-1259.)$ $\oplus 10^*(-1274.) \oplus \overline{10}^*(-1274.) \oplus 45(-1285.) \oplus 45(-1299.)$ $\oplus 55(-1322.) \oplus 45(-1323.) \oplus 44(-1326) \oplus 16(-1336)$ $\oplus \overline{16}(-1336) \oplus 10^*(-1357.) \oplus \overline{10}^*(-1357.) \oplus 45(-1365.)$ $\oplus 44(-1376) \oplus 55(-1380.) \oplus 55(-1425.)$	$45, 44$
$\left.\begin{matrix}8A\\8B\end{matrix}\right\}$	990	$1(920) \oplus 10(106.) \oplus 44(103.) \oplus 55(57.) \oplus 16(43.) \oplus \overline{16}(43.)$ $\oplus 45(31.) \oplus 55(28.) \oplus 44(25.) \oplus 55(24.) \oplus 44(20.) \oplus 10^*(14)$ $\oplus \overline{10}^*(14) \oplus 45(13.) \oplus 55(5.6) \oplus 10(3.1) \oplus 44(-0.6) \oplus 55(-8.)$ $\oplus 11(-10) \oplus 45(-14.) \oplus 16(-21.) \oplus \overline{16}(-21.) \oplus 55(-28.)$ $\oplus 44(-31.) \oplus 45(-35.) \oplus 44(-54.) \oplus 55(-64.) \oplus 45(-94.)$	$10^*, \overline{10^*}$
$\left.\begin{matrix}11A\\11B\end{matrix}\right\}$	720	$1(575) \oplus 45(96.) \oplus 44(79.) \oplus 55(75.) \oplus 11(35) \oplus 55(29.) \oplus 44(21.)$ $\oplus 45(11.) \oplus 55(0.7) \oplus 45(-7.3) \oplus 45(-7.6) \oplus 16(-10) \oplus \overline{16}(-10)$ $\oplus 55(-16.) \oplus 44(-43.) \oplus 45(-48.) \oplus 55(-64.) \oplus 44(-67.)$	$16, \overline{16}$

TABLE 2. Decomposition of span of conjugacy classes into irreps (eigenvalue of Killing form in brackets), for Matheiu group M_{11}. Classes are labelled by the order of elements and other symbols. Irreps are labelled by their dimensions and other symbols, with a bar for the conjugate representation. Non-integer eigenvalues are irrational and shown truncated with a period.

eigenvalues of the irreps occurring in the various conjugacy classes together as an overall picture or bar-code of the group, as well as an example of the contribution from just one class. Complex conjugates have the same spectrum so we do not list them and the thickness of lines indicates a preponderance of nearby eigenvalues. The combined spectrum is that of the direct sum D of all the different K acting on a Hilbert space $\mathcal{H}$ which is the direct sum of all the spans of conjugacy classes. The latter if we include the span of $\{e\}$ carries a faithful representation of the algebra of functions $\mathbb{C}(G)$ and the triple $(\mathbb{C}(G), \mathcal{H}, D)$ can be viewed as a discrete 'spectral triple' cf. [1] although obeying a more general set of axioms. This is a direction to be explored more fully in a sequel. One may compute

$$[f, D] = \sum_{g \in G \setminus \{e\}} e_g \partial^g f, \quad e_g(x_a) := \frac{|\mathcal{C}|}{|G|} K(x_a, x_{gag^{-1}}) x_{gag^{-1}}, \quad f x_a := f(a) x_a$$

for all $f \in \mathbb{C}(G)$ and $a \in \mathcal{C}$. Here

$$(\partial^g f)(a) := f(gag^{-1}) - f(a)$$

defines 'vector fields' for the conjugation action. We also have $f e_g = e_g f(g(\ )g^{-1})$ for the noncommutativity of 1-forms and functions. In this sense $[f, D]$ then has the form of an exterior differential with vielbein lengths given by K. The $|\mathcal{C}|/|G|$ weighting is to adjust for overcounting by the order of the isotropy group and could be omitted.

Finally, for completeness we mention that a more usual application of the Killing form in Lie theory is to the construction of the quadratic Casimir as a central element in the enveloping algebra. We can indeed do something similar whenever K is nondegenerate.

Definition 4.11. When the Killing form K is nondegenerate we let K^{ab} be its inverse matrix in basis $\mathcal{C}$. Then the quadratic Casimir is defined as $\tilde{C} = \sum_{a,b} K^{ab} x_a x_b \in U(\mathcal{L})$. Its image in the group algebra $C = \sum_{a,b} K^{ab} ab$ is also called the Casimir element.

Unlike K itself, C defines an operator not only on $\mathbb{C}\mathcal{C}$ but on every representation. We can of course use its values to induce a labelling of irreducibles. In the case of S_n this is hardly necessary but could be useful for other finite groups equipped with nondegenerate Killing form, much as in Lie theory. On the other hand we can also decompose $\mathcal{C}$ in terms of other 'θ elements' consisting of sums over other conjugacy classes. So this coordinatisation is a change of basis from using the various θ for different conjugacy classes, i.e. from using the so-called 'central characters' for the pairing of an irrep with a conjugacy class. Here the value of $\theta_{\mathcal{C}} = \sum_{a \in \mathcal{C}} x_a$ in an irrep V is given by

$$\theta_{\mathcal{C}}|_V = |\mathcal{C}| \frac{\chi_V(a)}{\chi_V(e)}, \quad a \in \mathcal{C}.$$

However, more geometrically, the action of the quadratic Casimir is the Laplace-Beltrami operator and this is analogous to the ability to view a classical quadratic differential as a 'linear' differential with respect to a different (noncommutative) differential calculus.

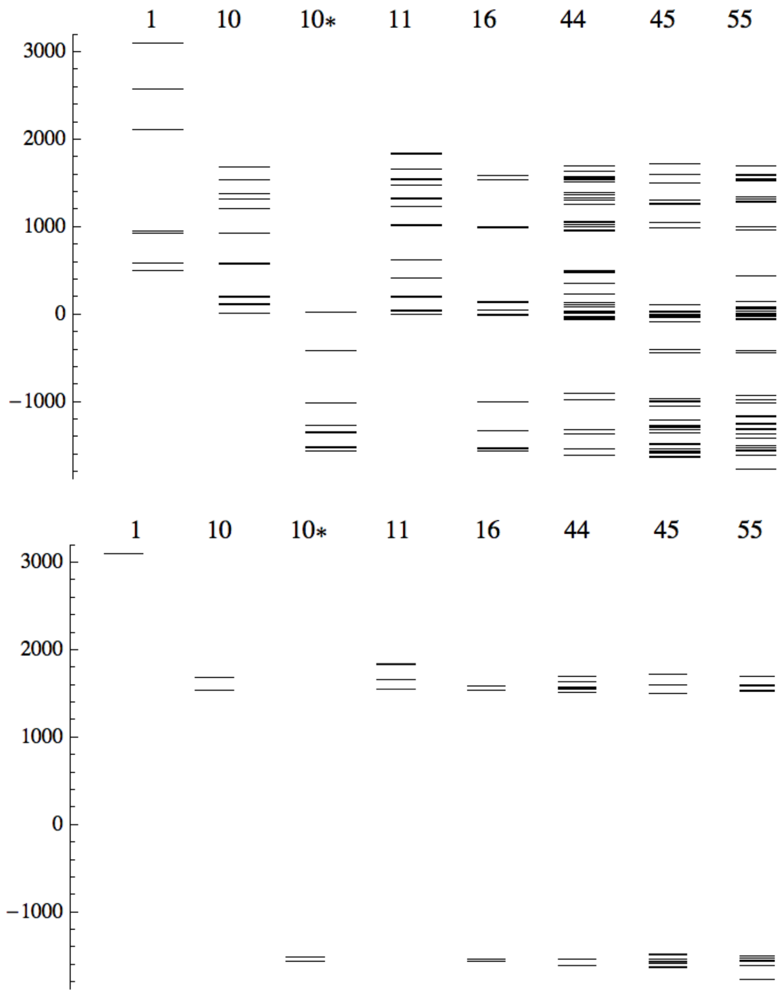


FIGURE 1. Killing forms spectrogram of Mathieu group M_{11} and below it of sample conjugacy class $5A$.

Example 4.12. For A_5 with its 2-2-cycles calculus we have K reducible into 5 identical 3×3 blocks K_i in a suitable basis order, with inverse of each block

$$K_i^{-1} = \frac{1}{84}\begin{pmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{pmatrix}.$$

From this one finds

$$C = \frac{15}{14}e - \frac{1}{42}\theta_{(12)(34)}$$

involving the same conjugacy class as defining the calculus. By contrast, for S_4 with its 2-cycles differential calculus we follow the above steps starting with the Killing form. This is also reducible and consists of 3 identical 2×2 blocks in a

suitable basis order with

$$K_i^{-1} = \frac{1}{16}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

and resulting quadratic Casimir

$$C = \frac{9}{8}e - \frac{1}{8}\theta_{(12)(34)}.$$

This is a combination of e and the θ for the 2-2-cycles conjugacy class and is more manifestly 'quadratic' for our 2-cycles calculus as expected for the natural Laplace-Beltrami operator on the finite group.

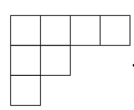
5. The Killing form and conjugation representation for S_n

Although we are primarily interested in simple groups, the symmetric groups are so sufficiently close and more readily computable that they are useful as a first indication. Also in noncommutative geometry S_3 in particularly with its 2-cycles class has the same noncommutative de Rham cohomology as $\mathbb{C}_q[SU_2]$ and has the flavour of some kind of discrete model of that. In this section we develop our theory for general S_n.

First we note that in the case of S_n for $n > 4$, with the 2-cycles conjugacy class, one can see from the formulae for the Killing form in [10] that K itself has all entries strictly positive. Hence the Proposition 4.4 applies in this case and there is a unique maximal eigenvalue, with eigenspace spanned by θ. For S_3 and S_4, K is reducible, and θ is a maximal eigenvector but each eigenvalue has multiplicity 3 (as one can see from Table 3 below). Our main result in this section is, for S_n with its 2-cycles class, an explicit decomposition of $\mathbb{C}\mathcal{C}$ into irreducible representations in a manner compatible with the eigenspace decomposition under K. We also find explicit formulae for the eigenvalues. As an application we obtain that the Killing form matrix K for this conjugacy class is positive definite.

Additionally at the end of the section we give a complete analysis for which conjugacy classes contain the sign representation.

First recall that irreducible representations of S_n, called the Specht modules, are indexed by partitions $\lambda \vdash n$, and a partition is represented by its *Young diagram* or *shape*. For example the partition $(4, 2, 1)$ of 7 is represented by



The Specht module associated to such a partition, or shape, is constructed abstractly for example in [4, 16].

We will use another construction [5] of the Specht module S^λ, namely as a subrepresentation of the regular representation. We recall this construction now. Note that, since S^λ occurs in the regular representation with multiplicity equal to $\dim S^\lambda$, the construction for given λ naturally depends on an additional choice. Namely we choose a *tableau* of shape λ, a one-to-one labelling of the boxes by the integers $\{1, \ldots, n\}$. As usual, the symmetric group S_n acts on the set of tableaux by permuting the entries.

To the chosen tableau T we associate its 'row subgroup' $R(T)$, consisting of permutations in S_n preserving the row sets, and its 'column subgroup' $C(T)$ of permutations preserving the column sets.

For example, for S_7, for the tableau T of shape $\lambda = (4, 2, 1)$ given by

$$\begin{array}{|c|c|c|c|}\hline 4 & 2 & 1 & 3 \\ \hline 5 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 7 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}\;,$$

the row sets are $\{1, 2, 3, 4\}, \{5, 6\}, \{7\}$ giving the product of permutation groups,

$$R(T) = Perm(\{1, 2, 3, 4\}) \times Perm(\{5, 6\}),$$

and the column sets are $\{4, 5, 7\}, \{2, 6\}, \{1\}, \{3\}$, so that

$$C(T) = Perm(\{4, 5, 7\}) \times Perm(\{2, 6\}).$$

To the tableau T we can then associate an element of the group algebra $\mathbb{C}S_n$ called the 'Young symmetrizer' $c_T = b_T a_T$ where

$$a_T = \sum_{\sigma \in R(T)} \sigma, \qquad b_T = \sum_{\sigma \in C(T)} (-1)^{\ell(\sigma)} \sigma$$

and $l(\sigma)$ is the length of a permutation. To finish the construction we consider the left ideal in the group algebra defined by the Young symmetrizer,

$$S^T := \mathbb{C}S_n \, c_T.$$

This defines a subrepresentation of the left regular representation. Clearly the right action of S_n on $\mathbb{C}S_n$ provides S_n-equivariant isomorphisms between the modules S^T for varying T. The isomorphism of any S^T with the abstract Specht module S^λ constructed in [4, 16] is straightforward to write down. Moreover, the sum of the subspaces S^T for varying tableaux T is the block of S^λ inside $\mathbb{C}S_n$.

Note that there are many more tableaux than the multiplicity of S^λ, so this is not a direct sum. Let $SYT(\lambda)$ denote the set of *standard Young tableaux*, that is tableaux whose entries are strictly increasing in rows and in columns. Then the block of S^λ inside $\mathbb{C}S_n$ is precisely the subspace

$$\bigoplus_{T \in SYT(\lambda)} S^T.$$

We now use this theory to give a concrete construction of irreducible subrepresentations inside a conjugation representation. For any partition $\mu = (\mu_1, \dots, \mu_k)$ of n we have a corresponding conjugacy class $\mathcal{C}_\mu$ in S_n, namely the one with cycle type μ. Explicitly $\mathcal{C}_\mu$ is the conjugacy class containing the element

$$a_\mu = (1, \dots, \mu_1)(\mu_1 + 1, \dots, \mu_1 + \mu_2) \dots (n - \mu_k + 1, \dots, n).$$

If we let Z_{a_μ} denote the centraliser of a_μ and identify $S_n / Z_{a_\mu} \cong \mathcal{C}_\mu$ via $\sigma Z_{a_\mu} \mapsto \sigma a_\mu \sigma^{-1}$, then we obtain a S_n-equivariant homomorphism from the left regular representation to the conjugation representation,

$$\pi : \mathbb{C}S_n \longrightarrow \mathbb{C}(S_n / Z_{a_\mu}) \cong \mathbb{C}\mathcal{C}_\mu, \tag{5.1}$$

coming from the linear extension of the quotient map $S_n \to S_n / Z_{a_\mu}$.

Lemma 5.1. *Suppose λ and μ are partitions of n and all notations as above.*

The Specht module S^λ occurs as a subrepresentation of the conjugation representation $\mathbb{C}\mathcal{C}_\mu$ if and only if there exists a standard Young tableau T of shape λ for which $c_T \cdot a_\mu \neq 0$ in $\mathbb{C}\mathcal{C}_\mu$.

In that case, the subrepresentation is explicitly realized as the subspace $\pi(S^T)$, where π is the projection map from (5.1).

Proof. If there is a tableau T for which $c_T \cdot a_\mu \neq 0$, then the restriction of the map π from (5.1) to the subrepresentation S^T of $\mathbb{C}S_n$ defines a non-zero S_n-equivariant map $S^T \to \mathcal{C}_\mu$. Since S^T is irreducible and isomorphic to S^λ it follows that this map must be an isomorphism onto its image, and therefore $\pi(S^T)$ is a submodule of the conjugation representation isomorphic to S^λ.

On the other hand, if $c_T \cdot a_\mu = 0$ for all $T \in SYT(\lambda)$, then the entire block of S^λ in $\mathbb{C}S_n$ lies in the kernel of π, and therefore the irreducible representation S^λ does not occur in the image of π. Since π is surjective, this means that S^λ is not a subrepresentation of $\mathbb{C}\mathcal{C}_\mu$. □

We remark that this lemma also holds, of course, with $\mathbb{C}\mathcal{C}_\mu$ replaced by any cyclic $\mathbb{C}S_n$-module, with the same proof.

Our first application of this construction is to the the 2-cycles class. In the example of S_3, the 2-cycles class $\mathcal{C}_{(2,1)}$ has three elements and it is straightforward to see that the conjugation representation, $\mathbb{C}\mathcal{C}_{(2,1)}$, is the (defining) three-dimensional permutation representation of S_3. In terms of Specht modules this representation decomposes as

$$\mathbb{C}\mathcal{C}_{(2,1)} = S^{(3)} \oplus S^{(2,1)}. \tag{5.2}$$

That is, the trivial representation plus the standard 2-dimensional representation. The general case is not much different, as we observe below.

We use the notation $(2, 1^{n-2})$ for the partition $(2, 1, \ldots, 1)$ which represents the 2-cycles class in S_n.

Proposition 5.2. *Consider S_n for $n > 2$ with the 2-cycles class $\mathcal{C} = \mathcal{C}_{(2,1^{n-2})}$. For $n = 3$ the decomposition of $\mathbb{C}\mathcal{C}$ into irreducibles is given in equation* (5.2).

(1) *For $n > 3$ the decomposition of the conjugation representation $\mathbb{C}\mathcal{C}$ into irreducible representations is given by*
$$\mathbb{C}\mathcal{C} \cong S^{(n)} \oplus S^{(n-1,1)} \oplus S^{(n-2,2)}.$$
Here the first two Specht modules $S^{(n)}, S^{(n-1,1)}$ are the trivial representation and the standard $(n-1)$-dimensional representation, respectively.

(2) *Each irreducible submodule of $\mathbb{C}\mathcal{C}$ lies in an eigenspace for the Killing form matrix K with eigenvalues as follows. The eigenvalue of K for the eigenspace containing $S^{(n)}$ (spanned by the element θ) is*
$$\frac{1}{4}(n^4 - 10n^3 + 41n^2 - 72n + 48).$$
The eigenvalue of K in the eigenspace containing $S^{(n-1,1)}$ is
$$n^2 - 6n + 12.$$

Suppose $n > 3$. Then the eigenvalue of K on the eigenspace containing $S^{(n-2,2)}$ is $2n$.

Proof. (1) could be checked using character theory. But we will rather define explicit embeddings of the Specht modules, by the method of Lemma 5.1, in order to be able to compute the eigenvalues of K in the later parts of the proof.

Of course the trivial representation embeds into $\mathbb{C}\mathcal{C}_{(2,1^{n-2})}$ as the subspace spanned by the element $\theta = \sum_{a\in\mathcal{C}} a$, and has multiplicity 1.

For the standard representation $S^{(n-1,1)}$ we consider the subspace $\pi(S^{T_1})$ of $\mathbb{C}\mathcal{C}_{(2,1^{n-2})}$ for π from (5.1) corresponding to the tableau

$$T_1 = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & \dots & n-1 \\ \hline n \\ \cline{1-1}\end{array}$$

This is the submodule of $\mathbb{C}\mathcal{C}$ obtained by applying $\mathbb{C}S_n$ to the vector $c_{T_1}\cdot(12)$. Up to an overall multiple, which we drop, this vector works out to be

$$v_{T_1} = (12) + (13) + \cdots + (1, n-1) - (2, n) - (3, n) - \cdots - (n-1, n).$$

Since $v_{T_1} \neq 0$ we have found a copy of $S^{(n-1,1)}$ in $\mathbb{C}\mathcal{C}$.

For the next representation $S^{(n-2,2)}$ we consider the subspace $\pi(S^{T_2})$ of $\mathbb{C}\mathcal{C}$ for π from (5.1) and the tableau

$$T_2 = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & \dots & n-2 \\ \hline \dots & n \\ \cline{1-2}\end{array}$$

This is the submodule of $\mathbb{C}\mathcal{C}$ obtained by applying $\mathbb{C}S_n$ to the vector $c_{T_2}\cdot(12)$. Up to an overall multiple this vector works out to be

$$v_{T_2} = (12) - (2, n-1) - (1, n) + (n-1, n),$$

and since $v_{T_2} \neq 0$ we have found a copy of $S^{(n-2,2)}$ in $\mathbb{C}\mathcal{C}$.

That we have thereby completely decomposed $\mathbb{C}\mathcal{C}$ follows by dimension count:

$$\dim S^{(n)} + \dim S^{(n-1,1)} + \dim S^{(n-2,2)} = 1 + (n-1) + \frac{n(n-3)}{2} = \binom{n}{2} = \dim \mathbb{C}\mathcal{C},$$

where $\dim(S^{(n-2,2)})$ is computed for example by the hook formula. (A prettier proof of the above sum decomposition for $\binom{n}{2}$ into dimensions of Specht modules goes via symmetric functions). This concludes the proof of (1).

That the irreducible subrepresentations lie in eigenspaces of K follows immediately from the fact that in the decomposition of $\mathbb{C}\mathcal{C}$ each irreducible representation occurs with multiplicity at most one. We can now compute the eigenvalues.

For the trivial representation we compute the column sum $\sum_a K((12), a)$ over all 2-cycles. In the basis of the 'triangular' listing

$$\begin{gathered}(12)\\ (13), (23)\\ (14), (24), (34)\\ (15), (25), (35), (45)\end{gathered}$$

$$\vdots \quad \vdots$$

$$(1n), (2n), (3n), (4n), \cdots, (n-1, n)$$

we have for a the choice (12), or a lies in the size $2(n-3)$ region on the left where a has one entry in common with (12), or a lies in the triangle to the right of size $(n-2)(n-3)/2$ where a is disjoint from (12). Using the values of K for these three cases in [10], we find

$$\binom{n}{2} + 2(n-2)\binom{n-3}{2} + \frac{(n-2)(n-3)}{2}\left(\binom{n-4}{2} + 2\right)$$

which computes as stated.

For the standard representation we use the vector we constructed in the proof of (1),

$$v_{T_1} = (12) + (13) + \cdots + (1, n-1) - (2, n) - (3, n) - \cdots - (n-1, n),$$

which involves the left and bottom slopes of the triangle leaving out the common vertex. Then the eigenvalue computed as the coefficient of (12) in $K(v_{T_1})$ is

$$\begin{aligned} &K((12),(12)) + (n-3)K((12),(13)) - K((12),(2,n)) - (n-3)K((12),(3,n)) \\ &\quad = \binom{n}{2} + (n-4)\binom{n-3}{2} + (n-3)\left(\binom{n-4}{2} + 2\right) \end{aligned}$$

which comes out as stated. Both formulae, although computed for $n > 4$ in the above counting, also give the right answer for $n = 2, 3, 4$, as computed by hand.

For the representation $S^{(n-2,2)}$ we use the vector

$$v_{T_2} = (12) - (2, n-1) - (1, n) + (n-1, n)$$

from the proof of (1) and compute the eigenvalue as the (12) coefficient of $K(v_{T_2})$, i.e. as

$$\begin{aligned} &K((12),(12)) - K((12),(2,n-1)) - K((12),(1,n)) + K((12),(n-1,n)) \\ &\quad = \binom{n}{2} - 2\binom{n-3}{2} + \binom{n-4}{2} + 2 = 2n. \end{aligned}$$

□

Corollary 5.3. *The Killing form for S_n, $n > 2$ with the 2-cycles conjugacy class $\mathcal{C}$ is non-degenerate and in fact positive definite. Moreover the decomposition of $\mathbb{C}\mathcal{C}$ into irreps consisting of the trivial and the standard representation, and the representation $S^{(n-2,2)}$, coincides for $n > 6$ with the decomposition of K into eigenspaces of respectively the maximal, next to maximal and smallest eigenvalues.*

Proof. Looking at the three expressions for the eigenvalues in the lemmas above it is evident that they have different leading powers of n and hence are distinct for all n bigger than some value. By inspection, the only degeneracies are $n = 3$ when the trivial and the standard representation have the same eigenvalue of K, $n = 4$ when the eigenvalues of the trivial and the $S^{(n-2,2)}$ coincide, being smaller than the eigenvalue of the standard representation, and $n = 6$ when the eigenvalues of the standard representation and of $S^{(n-2,2)}$ coincide. After that, the eigenvalue of the trivial exceeds that of the standard representation which exceeds that of $S^{(n-2,2)}$ as stated. As all the eigenvalues are positive we conclude that K is non-degenerate (and positive definite when extended as a hermitian inner product). □

One useful application of the K eigenvalue decomposition is that when it succeeds it directly provides subspaces of the conjugacy class representation which decompose it into irreps, just by linear algebra and entirely bypassing combinatorics. We illustrate this for S_7 and this also serves as a direct check of the above.

Example 5.4. For S_7 with its 2-cycles conjugacy class, we use the explicit formula for the Killing form in [10] and MATHEMATICA to compute the eigenvalues and eigenspaces. The eigenvalues are

$$\{131, 19, 19, 19, 19, 19, 19, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14, 14\}.$$

The unique 'Perron-Frobenius' eigenvector for the 131 eigenvalue is θ as it must be. The eigenvectors for the 19 eigenvalue are, with $x = -2/5, y = 3/5$,

$$e_1 = (y, y, x, y, x, x, y, x, x, x, y, x, x, x, x, 1, 0, 0, 0, 0, 0)$$

$$e_2 = (y, x, y, x, y, x, x, y, x, x, x, y, x, x, x, 0, 1, 0, 0, 0, 0)$$

$$e_3 = (x, y, y, x, x, y, x, x, y, x, x, x, y, x, x, 0, 0, 1, 0, 0, 0)$$

$$e_4 = (x, x, x, y, y, y, x, x, x, y, x, x, x, y, x, 0, 0, 0, 1, 0, 0)$$

$$e_5 = (x, x, x, x, x, x, y, y, y, y, x, x, x, x, y, 0, 0, 0, 0, 1, 0)$$

$$e_6 = (x, x, x, x, x, x, x, x, x, x, y, y, y, y, y, 0, 0, 0, 0, 0, 1)$$

in a basis ordered as above. On inspection we can see that conjugation by $(12), (23), (34), (45), (56)$ interchanges the e_i as in the standard representation namely $(12)e_1 = e_2, (12)e_2 = e_1$ and fixing the others, etc. The action of (67) is more complicated and comes out as $(67)e_i = e_i + v$ for $i = 1, \cdots, 5$ and $(67)e_6 = \sum_{i=1}^{6} e_i + v$ where

$$v = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -x, -x, -x, -x, -x, x, x, x, x, x, 0).$$

This obeys $(67)v = -v$. We now change to a new basis $\tilde{e}_i = e_i + \frac{v}{2}$ which does not change the form of the action of $(12), \cdots, (56)$ by transposition but now $(67)\tilde{e}_i = \tilde{e}_i$ for $i = 1, \cdots, 5$. We define $\tilde{e}_7 = (67)\tilde{e}_6$ to identify the representation as the standard one embedded in $\mathbb{C}^7$. The eigenvectors for the 14 eigenvalue can similarly be seen to form an irrep, the $S^{(n-2,2)}$ one.

Apart from accidental degeneracies and reversals for small n, we see that the next-to-maximal eigenvalue associates to the 2-cycles conjugacy class the standard representation, which is the Specht module associated to the transposition of the Young diagram describing the conjugacy class. Such an association also assigns the trivial representation to the trivial conjugacy class as expected. The methods we have developed are general and can be applied in principle to other conjugacy classes allowing us to study which if any irreps are picked out in a stable range with a view towards extending such a 1-1 correspondence. Such an extended correspondence, however, cannot be the obvious one because we find that not every Specht module is actually embeddable in the adjoint representation of the 'expected' conjugacy class under the association given by transposition of the Young diagram. Specifically, we show in Section 5.1 that the sign representation does not occur in the n-cycle class for all even n. At least for small n we can see, however, that an extended correspondence is possible.

Example 5.5. A complete analysis for S_3, S_4 using methods as above is summarised in Table 1. One could obtain these results from Sage but here we obtain them as an illustration of the general theory above to conjugacy classes of different shape. We focus on S_4 and we first realise the relevant Specht modules S^λ as submodules of the left regular representation $\mathbb{C}S_4$ explicitly as

$$S^{(4)} = \ydiagram{4} \cong \mathbb{C}S_4\, c_{\ytableaushort{1234}},\quad S^{(3,1)} = \ydiagram{3,1} \cong \mathbb{C}S_4\, c_{\ytableaushort{123,4}},\quad S^{(2,2)} = \ydiagram{2,2} \cong \mathbb{C}S_4\, c_{\ytableaushort{12,34}}$$

$$S^{(2,1,1)} = \ydiagram{2,1,1} \cong \mathbb{C}S_4\, c_{\ytableaushort{14,2,3}},\quad S^{(1,1,1,1)} = \ydiagram{1,1,1,1} \cong \mathbb{C}S_4\, c_{\ytableaushort{1,2,3,4}}$$

These give rise to explicit embeddings of these Specht modules into the conjugation representation as follows.

(1) For the $(2,2)$-cycle conjugacy class in S_4 we have an explicit embedding of the Specht module $S^{(2,2)}$ given by applying $\mathbb{C}S_n$ to the vector

$$c_{\ytableaushort{12,34}} \cdot (12)(34) = 8\left((12)(34) - (23)(14)\right).$$

As this vector is non-zero, this 2-dimensional representation is contained and by adding up the dimensions

$$\mathbb{C}\mathcal{C}_{(2,2)} = \ydiagram{4} \oplus \ydiagram{2,2}.$$

Next, the Killing form matrix has all entries 3 because the product of any 2-2-cycles here is another 2-2-cycle and all elements of the commute. Hence $K((12)(34), v)) = 0$ so this module has zero eigenvalue.

(2) For the 3-cycle conjugacy class in S_4 we similarly find

$$\mathbb{C}\mathcal{C}_{(3,1)} = \ydiagram{4} \oplus \ydiagram{3,1} \oplus \ydiagram{2,1,1} \oplus \ydiagram{1,1,1,1},$$

with explicit embeddings of the Specht modules $S^{(3,1)}, S^{(2,1,1)}$ and $S^{(1^4)}$ given by applying $\mathbb{C}S_n$ to the non-zero vectors

$$c_{\ytableaushort{123,4}} \cdot (123) = 6\left((123) + (132) - (234) - (243)\right)$$

$$c_{\ytableaushort{14,2,3}} \cdot (123) = 3(123) - 3(132) + (124) - (142) + (143) - (134) + (234) - (243)$$

$$c_{\ytableaushort{1,2,3,4}} \cdot (123) = 3\left((123) - (132) - (234) + (243) + (134) - (143) - (124) + (142)\right).$$

Next, all products of 2-cycles are either e, a 3-cycle or a 2-2-cycle and we compute the number of fixed points (the character of the conjugation representation) as $8, 2, 0$ respectively. Hence the Killing form in basis order

$(123), (132), (142), (124), (134), (143), (243), (234)$ comes out as

$$K_{(3,1)} = \begin{pmatrix} 2&8&2&0&2&0&2&0\\ 8&2&0&2&0&2&0&2\\ 2&0&2&8&2&0&2&0\\ 0&2&8&2&0&2&0&2\\ 2&0&2&0&2&8&2&0\\ 0&2&0&2&8&2&0&2\\ 2&0&2&0&2&0&2&8\\ 0&2&0&2&0&2&8&2 \end{pmatrix}.$$

Denoting by v the stated eigenvector taken without any overall factor, we compute the eigenvalue from the coefficient of (123) in $K((123), v)$. For example, in the first case $K((123), v) = 2 + 8 - 0 - 2 = 8$.

(3) For the 4-cycle conjugacy class in S_4 we similarly find

$$\mathbb{C}\mathcal{C}_{(4)} = \ydiagram{4} \oplus \ydiagram{2,2} \oplus \ydiagram{2,1,1},$$

with the explicit embeddings of the Specht modules $S^{2,2}$ and $S^{2,1,1}$ given by applying $\mathbb{C}S_n$ to the non-zero vectors

$$c_{\ytableaushort{12,34}} \cdot (1234) = 2\left((1342) + (1243) - (1423) - (1324)\right)$$

$$c_{\ytableaushort{14,2,3}} \cdot (1234) = 2\left((1234) + (1243) + (1423) - (1342) - (1324) - (1432)\right).$$

This time products of 4-cycles are either e, 2-2-cycles or 3-cycles with number of fixed points or character values 6,2,0 respectively. Hence the Killing form in basis order $(1234), (1243), (1324), (1342), (1423), (1432)$ is

$$K_{(4)} = \begin{pmatrix} 2&0&0&0&0&6\\ 0&2&0&6&0&0\\ 0&0&2&0&6&0\\ 0&6&0&2&0&0\\ 0&0&6&0&2&0\\ 6&0&0&0&0&2 \end{pmatrix}.$$

We use the eigenvectors v above without the overall factors and compute the eigenvalue as the coefficient of (1243) in $K((1243), v)$. For example, in the first case $K((1243), v) = 6 + 2 - 0 - 0 = 6$.

Looking as the tables we see that for S_3 the 'expected' correspondence holds but the eigenvalues of K are degenerate in the case of 2-cycles (as above). For S_4 there are also some eigenvalue degeneracies and the 4-cycle representation does not contain the 'expected' Specht module $\bar{1}$. On the other hand, we see that a different assignment of irreps to conjugacy classes is possible in such a way as to give a 1-1 correspondence between irreps and conjugacy classes.

We end with a couple of more concrete conjectures for S_n based on limited computer verification to $n \leq 8$. They remain a topic for further work.

Conjecture 5.6. For S_n, $n > 4$ the conjugacy classes with reducible K are precisely the $\frac{n-1}{2}$-fold 2-cycles for n odd. In this case the maximal eigenvalue has eigenspace decomposition $1 \oplus (n-1)$, where $n-1$ is the standard representation.

$\mathcal{C} \subset S_3$	$\lvert\mathcal{C}\rvert$	Decomposition (eigenvalue)	preferred irrep
(12)	3	$1(3) \oplus 2(3)$	2
(123)	2	$1(9) \oplus \bar{1}(0)$	$\bar{1}$

$\mathcal{C} \subset S_4$	$\lvert\mathcal{C}\rvert$	Decomposition (eigenvalue)	preferred irrep
(12)	6	$1(8) \oplus 2(8) \oplus 3(4)$	3
(12)(34)	3	$1(9) \oplus 2(0)$	2
(123)	8	$1(16) \oplus 3(8) \oplus \bar{1}(0) \oplus \bar{3}(-8)$	$\bar{1}$
(1234)	6	$1(8) \oplus 2(8) \oplus \bar{3}(-4)$	$\bar{3}$

TABLE 3. Decomposition of span of conjugacy classes into irreps (eigenvalue of Killing form in brackets), for S_3, S_4. Here irreps are labelled by their dimensions with a bar when tensored with the sign representation.

Conjecture 5.7. For all S_n, $n > 4$ all conjugacy classes have non-degenerate K and of these precisely $[\frac{n}{2}]$ are positive definite, namely the classes of m-fold 2-cycles class for $m = 1, 2, \cdots, [\frac{n}{2}]$. Here [ ] denotes integer part. All the other classes have evenly split signature.

Note that of the positive-definite classes, the 2-cycles remain the more natural one for the 'compact real form' of the braided Lie algebra of S_n, with the others viewed as defining higher order differential calculi.

5.1. **The sign representation in the conjugation representation.** Not much is known about decomposition of the conjugation representation of S_n into irreducible representations other than Roth's property, that every irreducible representation occurs [3, 17] if $n > 2$, and some asymptotic estimates for the multiplicities [14]. The more detailed question of which conjugacy classes a given irreducible representation occurs in looks even more complicated in general. However for the sign representation we can give a complete answer, which we do here.

First we explain briefly the overall multiplicity of the sign representation in the conjugation representation $\mathbb{C}S_n$. By character theory this multiplicity is precisely the number of conjugacy classes consisting of even permutations minus the number of conjugacy classes of odd permutations (the row sum in the character table, for the sign representation). In analogy with Euler's product formula for the number of all partitions, one can write a generating function for these multiplicities.

Namely, if $s(n)$ is the multiplicity of the sign representation in $\mathbb{C}S_n$, then the above description of $s(n)$ implies

$$1 + t + \sum_{n=2}^{\infty} s(n)t^n = \prod_{k=1}^{\infty} \left(\frac{1}{1 + (-t)^k} \right). \tag{5.3}$$

Clearly, multiplying out the right-hand side of the formula above, we get a contribution ± 1 to t^n for every partition of n. Namely the contribution is $+1$ for every partition of n with an even number of even parts and -1 for every partition of n with an odd number of even parts. The former are in bijection with conjugacy

classes of even elements, and the latter with conjugacy classes of odd elements. This proves the identity (5.3).

By the classical Euler identity which reads (after replacing the usual variable by $-t$ and inverting),

$$\prod_{k=1}^{\infty}\left(\frac{1}{1+(-t)^k}\right)=\prod_{k=1}^{\infty}(1+t^{2k-1}), \tag{5.4}$$

it follows that the multiplicity of the sign representation in the conjugation representation $\mathbb{C}S_n$ is equal to the number of partitions of n into distinct odd parts. We refine this observation in the following proposition, which could alternatively be viewed as giving a representation theoretic proof of the Euler identity (5.4).

Proposition 5.8. *The sign representation of S_n appears as a subrepresentation of the conjugation representation $\mathbb{C}\mathcal{C}_\mu$ if and only if μ is a partition of n into distinct odd parts. If it appears in $\mathbb{C}\mathcal{C}_\mu$, then it has multiplicity one.*

Proof. Since the sign representation has multiplicity one in in the left-regular representation $\mathbb{C}S_n$ and the conjugation representation $\mathbb{C}\mathcal{C}_\mu$ is a cyclic $\mathbb{C}S_n$-module, it is clear that the sign representation can have multiplicity at most 1 in $\mathbb{C}\mathcal{C}_\mu$.

Let us now write $\sigma\cdot\sigma'=\sigma\sigma'\sigma^{-1}$ for the conjugation action. Fix an element a_μ in the conjugacy class $\mathcal{C}_\mu$. By Lemma 5.1, the sign representation appears in $\mathbb{C}\mathcal{C}_\mu$ if and only if the element

$$v_\mu=\sum_\sigma(-1)^{\ell(\sigma)}\sigma\cdot a_\mu$$

in $\mathbb{C}\mathcal{C}_\mu$ is nonzero. Moreover if it is nonzero then it spans the sign representation. Now suppose v_μ is nonzero and let τ be an element of the centraliser Z_{a_μ}. Then we see that

$$\tau\ \cdot\ v_\mu\ =\ \sum_\sigma(-1)^{\ell(\sigma)}(\tau\sigma\tau^{-1})\tau\ \cdot\ a_\mu\ =\ \sum_\sigma(-1)^{\ell(\sigma)}\tau\sigma\tau^{-1}\ \cdot\ a_\mu\ =\ v_\mu.$$

This implies that τ is even, since v_μ spans the sign representation. Therefore if the sign representation occurs in $\mathbb{C}\mathcal{C}_\mu$ then Z_{a_μ} contains only even permutations.

The converse is true as well. If all elements in Z_μ are even, then the coefficient of a_μ in v_μ comes out to be $|Z_\mu|$, implying that v_μ is nonzero, and the sign representation occurs in $\mathbb{C}C_\mu$.

It remains to prove that Z_{a_μ} contains only even permutations, precisely if μ is a permutation of n into distinct odd parts.

Clearly, if μ has an even part then there is a cycle of even length in a_μ, which gives an element of the centralizer that has odd parity. Also if μ has two parts of size k (we may assume k odd, by above), then there is an element of the centralizer which exchanges the corresponding two k-cycles of a_μ, which is a product of k many 2-cycles. So again there is an element of odd parity in Z_{a_μ}. This shows that for the sign representation to occur inside $\mathbb{C}\mathcal{C}_\mu$, we must have that μ is a partition of n into distinct, odd parts.

Conversely, if μ is a partition of n into distinct odd parts, then the centraliser is generated by the individual cycles in a_μ, and these are all even permutations. $\square$

We note that since, as in the proof of the previous proposition, one can always give a precise vector spanning a copy of the sign representation, one can in principle use the methods from above to also find its eigenvalue under K.

As a very special case, note that Proposition 5.8 implies in particular that if n is even, then the sign representation does not occur in the class of n-cycles. This means that, as already remarked, the naive guess at a bijection of conjugacy classes with Specht modules contained within them (given by transposing the diagram) does not work in general. Nevertheless, for the cases we have looked there *is* at least one alternative assignment. Thus for S_n with $n > 2$ it is quite possible that one should be able to find each irrep in a distinct conjugacy class, but clearly some further ideas are needed.

6. Computer verifications for simple groups

In order to provide empirical supporting evidence for our conjectures as well as get a grip on the behaviour of the Killing forms associated to minimal calculi for finite simple groups we have performed an extensive amount of computational verifications.

Most of our calculations have been performed using the open source computer algebra system Sage in a linux workstation, relying heavily on GAP for some internal procedures. The code of the actual implementation is available from the authors upon request.

In the present section we list the computational methods chosen as well as the obtained results.

6.1. **Effective calculation of the Killing form.** In order to compute the Killing form K associated to a conjugacy class $\mathcal{C} = g^G$ we take advantage of the ad-invariance $K(x_{aga^{-1}}, x_h) = K(x_g, x_{a^{-1}ha})$. A sketch of the procedure goes as follow:

STEP 1: Pick a generator g of $\mathcal{C}$,
STEP 2: Compute a section $s : \mathcal{C} \to G$, such that $h = s(h)gs(h)^{-1}$ for all $h \in \mathcal{C}$,
STEP 3: Compute the function $f(h) = |Z(gh) \cap \mathcal{C}|$ and cache its values,
STEP 4: Compute K_{ab} as $f(s(a)^{-1}bs(a))$.

In practical terms, the suggested method reduces the computation of the Killing form to the computation of the first row and the permutations that create all the other rows from the first one. The resulting algorithm provides a reasonably quick computation of the Killing form, allowing to compute them for all conjugacy classes of all finite simple groups of order up to 75000. The limiting factors of the implementation are related to exhaustion of the computer memory rather than computing times, with the first conjugacy class out of our reach being the class 6B of elements of order 6 with centralizer of size 6 in the next larger group, the Mathieu group M_{12} of order 95040. The naming of the conjugacy classes is according to the convention in the ATLAS [2].

6.2. **Exploring nondegeneracy of K.** Nondegeneracy of K is checked by using Sage internal algorithms for computing the rank of K. Computationally, finding the rank is quick enough and doesn't require substantially more resources than computing the matrix itself. Up to order 75,000 the Killing form is nondegenerate in all cases but the following exceptions:

- The (mutually inverse) conjugacy classes 7A and 7B of elements of order 7 in the alternating group A_7,
- The (pairwise inverse) conjugacy classes 4A, 4B, 8A, 8B, 12A, 12B of elements of orders 4, 8 or 12 in the unitary group $PSU_3(3) = G_2(2)'$.
- The two (mutually inverse) conjugacy classes 7A and 7B of elements of order 7 in $PSL(3, 4)$.

As mentioned before, all the degenerate cases occur in conjugacy classes that are not closed under inversion, so reality of the conjugacy class appears to be a sufficient condition (though by no means necessary) for the nondegeneracy of K.

6.3. **Irreducibility of K.** The irreducibility of K is tested by using standard algorithms for checking connectedness of the graph G_K with vertices indexed by elements of $\mathcal{C}$ and containing an edge (a, b) if and only if $K_{a,b} \neq 0$. The size of the graph G_K is substantially smaller than the size of K and the connectedness checking is also fast, so again this test does not produce much of an additional overhead.

Generically, the tested Killing forms yield irreducible matrices, so that Perron-Frobenius theorem applies and the eigenspace associated to the maximal eigenvalue is 1-dimensional; the only observed exceptions to irreducibility are given by the conjugacy classes of involutions in the linear groups $PSL(2, 4) = A_5$, $PSL(2, 8)$, $PSL(2, 16)$, $PSL(2, 32)$, the exceptional Suzuki group $Suz(8)$ and the unitary group $PSU(3, 4)$. The observed reducible cases suggest a particular behaviour for the classes of involutions only when the group naturally appears as a group of matrices with coefficients on a field of characteristic 2. However, not every such group and class of involutions is reducible, as shown by the data for $PSL(3, 4)$.

6.4. **Eigenspaces and irrep decompositions.** For this problem we find some additional limiting factors. The computation of the characteristic polynomial and the determination of the eigenvalues gets slow and as the size of the conjugacy classes increase. The eigenvalues with their corresponding multiplicity have been computed for all the listed groups. That computation has revealed that the Killing form appears to be positive definite whenever it comes from a conjugacy class consisting of involutions, plus the mutually inverse classes 3A and 3B of elements of order 3 and centralizer of size 648 in the unitary group $PSU(4, 2)$. Thus to the level visible in the data and for real conjugacy classes the classes of involutions are precisely those where the Killing form is positive definite. Moreover, we see also that for real conjugacy classes those which are not classes of involutions have precisely an equal number of positive and negative eigenvalues taken with multiplicity, i.e. zero signature.

For the same groups, we have also computed the decomposition into irreps of the adjoint representation on $\mathbb{C}\mathcal{C}$ by means of character theory, looking for some correlation between both decompositions. As the group get larger, the observed behaviour is that the dimensions of the eigenspaces coincide with the dimensions of irreducible representations, so as the group size increases we expect that each eigenspace contains exactly one irrep. The obvious exceptions to this rule are the conjugacy classes yielding reducible Killing forms mentioned in the previous paragraph.

We have tried to make the correlation between irreps and eigenspaces more precise, and implemented an algorithm that finds the irrep decomposition of each eigenspace. However, the exact determination of the eigenspaces turns out to be too slow and memory intensive to be of any practical use except for the few smallest conjugacy classes. We hope we will be able to get around this issue in the near future.

6.5. **Data:** Here we summarize some of the obtained data for all finite simple groups up to order 75,000. We list whether the conjugacy class is real (closed for inverses), whether the corresponding Killing form irreducible, and its signature. The naming of the conjugacy classes follows the convention at the ATLAS, and conjugacy classes of elements order with the same centralizer sizes have been amalgamated whenever they show identical behaviour. Listing the actual eigenspace decomposition of the adjoint representation on $\mathbb{C}\mathcal{C}$ would be too lengthy and not particularly enlightening, so we shall omit that data here. Whenever the Killing form is reducible we have included between brackets the number of irreducible components in the corresponding column. The signature of the Killing form is expressed as the triple (p, n, z) where p, n and z are respectively the number (counted with multiplicities) of positive, negative and zero eigenvalues; in particular, nondegeneracy is given by zero as the last number of this triple. In supplementary information we list the maximal eigenvalue λ_{max} of the Killing form, equal to the row sum. For a real conjugacy class $(\lambda_{max} - |\mathcal{C}|)/|\mathcal{C}|$ is a measure of the typical size of the other entries of the Killing form matrix after the principal entry $|\mathcal{C}|$ in each row. We also list the value $\chi_{\mathcal{C}}(\mathcal{C})$ of the character of the adjoint representation on a typical element of $\mathcal{C}$ as a measure of the degree to which the braided Lie algebra is nonabelian. It counts the number of elements in $\mathcal{C}$ that commute with any given element of $\mathcal{C}$.

A_5, order 60

$\mathcal{C}$	$\lvert\mathcal{C}\rvert$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	15	3	True	False (5)	21	(15, 0, 0)
$3A$	20	2	True	True	34	(10, 10, 0)
$5A - B$	12	2	True	True	24	(6, 6, 0)

$PSL(2,7)$, order 168

$\mathcal{C}$	$\lvert\mathcal{C}\rvert$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	21	5	True	True	49	(21, 0, 0)
$3A$	56	2	True	True	94	(28, 28, 0)
$4A$	42	2	True	True	76	(21, 21, 0)
$8A - B$	24	3	False	True	30	(16, 8, 0)

A_6, order 360

$\mathcal{C}$	$\|\mathcal{C}\|$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	45	5	True	True	73	(45, 0, 0)
$3A-B$	40	4	True	True	88	(20, 20, 0)
$4A$	90	2	True	True	156	(45, 45, 0)
$5A-B$	72	2	True	True	134	(36, 36, 0)

$PSL(2,8)$, order 504

$\mathcal{C}$	$\|\mathcal{C}\|$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	63	7	True	False (9)	105	(63, 0, 0)
$3A$	56	2	True	True	112	(28, 28, 0)
$7A-C$	72	2	True	True	130	(36, 36, 0)
$9A-C$	56	2	True	True	112	(28, 28, 0)

$PSL(2,11)$, order 660

$\mathcal{C}$	$\|\mathcal{C}\|$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	55	7	True	True	121	(55, 0, 0)
$3A$	110	2	True	True	208	(55, 55, 0)
$5A-B$	132	2	True	True	234	(66, 66, 0)
$6A$	110	2	True	True	208	(55, 55, 0)
$11A-B$	60	5	False	True	80	(36, 24, 0)

$PSL(2,13)$, order 1092

$\mathcal{C}$	$\|\mathcal{C}\|$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	91	7	True	True	157	(91, 0, 0)
$3A$	182	2	True	True	328	(91, 91, 0)
$6A$	182	2	True	True	328	(91, 91, 0)
$7A-C$	156	2	True	True	298	(78, 78, 0)
$13A-B$	84	6	True	True	192	(42, 42, 0)

$PSL(2,17)$, order 2448

$\mathcal{C}$	$\|\mathcal{C}\|$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	153	9	True	True	273	(153, 0, 0)
$3A$	272	2	True	True	526	(136, 136, 0)
$4A$	306	2	True	True	564	(153, 153, 0)
$8A-B$	306	2	True	True	564	(153, 153, 0)
$9A-C$	272	2	True	True	526	(136, 136, 0)
$17A-B$	144	8	True	True	336	(72, 72, 0)

A_7, order 2520

$\mathcal{C}$	$\|\mathcal{C}\|$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	105	9	True	True	273	(105, 0, 0)
$3A$	70	10	True	True	256	(35, 35, 0)
$3B$	280	4	True	True	616	(140, 140, 0)
$4A$	630	2	True	True	1068	(315, 315, 0)
$5A$	504	4	True	True	936	(252, 252, 0)
$6A$	210	6	True	True	528	(105, 105, 0)
$7A-B$	360	3	False	True	324	(171, 140, 49)

$PSL(2,19)$, order 3420

$\mathcal{C}$	$\lvert\mathcal{C}\rvert$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	171	11	True	True	361	(171, 0, 0)
$3A$	380	2	True	True	706	(190, 190, 0)
$5A-B$	342	2	True	True	664	(171, 171, 0)
$9A-C$	380	2	True	True	664	(190, 190, 0)
$10A-B$	342	2	True	True	706	(171, 171, 0)
$19A-B$	180	9	False	True	252	(100, 80, 0)

$PSL(2,16)$, order 4080

$\mathcal{C}$	$\lvert\mathcal{C}\rvert$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	255	15	True	False (17)	465	(255, 0, 0)
$3A$	272	2	True	True	514	(136, 136, 0)
$5A-B$	272	2	True	True	514	(136, 136, 0)
$15A-D$	272	2	True	True	514	(136, 136, 0)
$17A-H$	240	2	True	True	480	(120, 120, 0)

$PSL(3,3)$, order 5616

$\mathcal{C}$	$\lvert\mathcal{C}\rvert$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	117	13	True	True	489	(117, 0, 0)
$3A$	104	14	True	True	412	(52, 52, 0)
$3B$	624	6	True	True	1224	(312, 312, 0)
$4A$	702	2	True	True	1356	(351, 351, 0)
$6A$	936	2	True	True	1848	(468, 468, 0)
$8A-B$	702	2	False	True	600	(337, 365, 0)
$13A-B$	432	3	False	True	399	(224, 208, 0)
$13C-D$	432	3	False	True	399	(236, 196, 0)

$PSU(3,3) \cong G_2^{2\prime}$, order 6048

$\mathcal{C}$	$\lvert\mathcal{C}\rvert$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	63	7	True	True	177	(63, 0, 0)
$3A$	56	2	True	True	112	(28, 28, 0)
$3B$	672	6	True	True	1332	(336, 336, 0)
$4A-B$	63	7	False	True	105	(22, 14, 27)
$4C$	378	6	True	True	852	(189, 189, 0)
$6A$	504	2	True	True	1104	(252, 252, 0)
$7A-B$	864	3	False	True	555	(436, 428, 0)
$8A-B$	756	2	False	True	752	(364, 365, 27)
$12A-B$	504	2	False	True	480	(238, 224, 42)

$PSL(2,23)$, order 6072

$\mathcal{C}$	$\lvert\mathcal{C}\rvert$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	253	13	True	True	529	(253, 0, 0)
$3A$	506	2	True	True	988	(253, 253, 0)
$4A$	506	2	True	True	988	(253, 253, 0)
$6A$	506	2	True	True	988	(253, 253, 0)
$11A-E$	552	2	True	True	1038	(276, 276, 0)
$12A-B$	506	2	True	True	988	(253, 253, 0)
$23A-B$	264	11	False	True	374	(144, 120, 0)

$PSL(2,25)$, order 7800

$\mathcal{C}$	$\lvert\mathcal{C}\rvert$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	325	13	True	True	601	(325, 0, 0)
$3A$	650	2	True	True	1228	(325, 325, 0)
$4A$	650	2	True	True	1228	(325, 325, 0)
$5A-B$	312	12	True	True	744	(156, 156, 0)
$6A$	650	2	True	True	1228	(325, 325, 0)
$12A-B$	650	2	True	True	1228	(325, 325, 0)
$13A-F$	600	2	True	True	1174	(300, 300, 0)

M_{11}, order 7920

$\mathcal{C}$	$\lvert\mathcal{C}\rvert$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	165	13	True	True	489	(165, 0, 0)
$3A$	440	8	True	True	946	(220, 220, 0)
$4A$	990	2	True	True	2108	(495, 495, 0)
$5A$	1584	4	True	True	3096	(792, 792, 0)
$6A$	1320	2	True	True	2568	(660, 660, 0)
$8A-B$	990	2	False	True	920	(515, 475, 0)
$11A-B$	720	5	False	True	575	(355, 365, 0)

$PSL(2,27)$, order 9828

$\mathcal{C}$	$\lvert\mathcal{C}\rvert$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	351	15	True	True	729	(351, 0, 0)
$3A-B$	364	13	False	True	520	(196, 168, 0)
$7A-C$	702	2	True	True	1376	(351, 351, 0)
$13A-F$	756	2	True	True	1434	(378, 378, 0)
$14A-C$	702	2	True	True	1376	(351, 351, 0)

$PSL(2,29)$, order 12180

$\mathcal{C}$	$\lvert\mathcal{C}\rvert$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	435	15	True	True	813	(435, 0, 0)
$3A$	812	2	True	True	1594	(406, 406, 0)
$5A-B$	812	2	True	True	1594	(406, 406, 0)
$7A-C$	870	2	True	True	1656	(435, 435, 0)
$14A-C$	870	2	True	True	1656	(435, 435, 0)
$15A-D$	812	2	True	True	1594	(406, 406, 0)
$29A-B$	420	14	True	True	1008	(210, 210, 0)

$PSL(2,31)$, order 14880

$\mathcal{C}$	$\lvert\mathcal{C}\rvert$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	465	17	True	True	961	(465, 0, 0)
$3A$	992	2	True	True	1894	(496, 496, 0)
$4A$	930	2	True	True	1828	(465, 465, 0)
$5A-B$	992	2	True	True	1894	(496, 496, 0)
$8A$	930	2	True	True	1828	(465, 465, 0)
$15A-D$	992	2	True	True	1894	(496, 496, 0)
$16A-E$	930	2	True	True	1828	(465, 465, 0)
$31A-B$	480	15	False	True	690	(256, 224, 0)

A_8, order 20160

$\mathcal{C}$	$\|\mathcal{C}\|$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	105	25	True	True	849	(105, 0, 0)
$2B$	210	18	True	True	996	(210, 0, 0)
$3A$	112	22	True	True	784	(56, 56, 0)
$3B$	1120	4	True	True	3028	(560, 560, 0)
$4A$	1260	8	True	True	3280	(630, 630, 0)
$4B$	2520	4	True	True	4736	(1260, 1260, 0)
$5A$	1344	4	True	True	2996	(672, 672, 0)
$6A$	1680	6	True	True	3600	(840, 840, 0)
$6B$	3360	2	True	True	6168	(1680, 1680, 0)
$7A-B$	2880	3	False	True	2466	(1375, 1505, 0)
$15A-B$	1344	4	False	True	1556	(597, 747, 0)

$PSL(3,4)$, order 20160

$\mathcal{C}$	$\|\mathcal{C}\|$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	315	27	True	True	1305	(315, 0, 0)
$3A$	2240	8	True	True	4888	(1120, 1120, 0)
$4A-C$	1260	12	True	True	3312	(630, 630, 0)
$5A-B$	4032	2	True	True	7284	(2016, 2016, 0)
$7A-B$	2880	3	False	True	2466	(1398, 1302, 180)

$PSL(2,37)$, order 25308

$\mathcal{C}$	$\|\mathcal{C}\|$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	703	19	True	True	1333	(703, 0, 0)
$3A$	1406	2	True	True	2704	(703, 703, 0)
$6A$	1406	2	True	True	2704	(703, 703, 0)
$9A-C$	1406	2	True	True	2704	(703, 703, 0)
$18A-C$	1406	2	True	True	2704	(703, 703, 0)
$19A-I$	1332	2	True	True	2626	(666, 666, 0)
$37A-B$	684	18	True	True	1656	(342, 342, 0)

$PSU(4,2)$, order 25920

$\mathcal{C}$	$\|\mathcal{C}\|$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	45	13	True	True	201	(45, 0, 0)
$2B$	270	22	True	True	1188	(270, 0, 0)
$3A-B$	40	13	False	True	196	(40, 0, 0)
$3C$	240	6	True	True	720	(120, 120, 0)
$3D$	480	12	True	True	1548	(240, 240, 0)
$4A$	540	8	True	True	1488	(270, 270, 0)
$4B$	3240	4	True	True	5440	(1620, 1620, 0)
$5A$	5184	4	True	True	9836	(2592, 2592, 0)
$6A-B$	360	5	False	True	708	(231, 129, 0)
$6C-D$	720	4	False	True	1272	(364, 356, 0)
$6E$	1440	2	True	True	3336	(720, 720, 0)
$6F$	2160	2	True	True	4176	(1080, 1080, 0)
$9A-B$	2880	3	False	True	2646	(1595, 1285, 0)
$12A-B$	2160	2	False	True	1824	(1035, 1125, 0)

Suz_8, order 29120

$\mathcal{C}$	$\|\mathcal{C}\|$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	455	7	True	False (65)	497	(455, 0, 0)
$4A-B$	1820	4	False	True	2768	(755, 1065, 0)
$5A$	5824	4	True	True	9796	(2912, 2912, 0)
$7A-C$	4160	2	True	True	7690	(2080, 2080, 0)
$13A-C$	2240	4	True	True	4748	(1120, 1120, 0)

$PSL(2,32)$, order 32736

$\mathcal{C}$	$\|\mathcal{C}\|$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	1023	31	True	False (33)	1953	(1023, 0, 0)
$3A$	992	2	True	True	1984	(496, 496, 0)
$11A-E$	992	2	True	True	1984	(496, 496, 0)
$31A-O$	1056	2	True	True	2050	(528, 528, 0)
$33A-J$	992	2	True	True	1984	(496, 496, 0)

$PSL(2,41)$, order 34440

$\mathcal{C}$	$\|\mathcal{C}\|$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	861	21	True	True	1641	(861, 0, 0)
$3A$	1640	2	True	True	3238	(820, 820, 0)
$4A$	1722	2	True	True	3324	(861, 861, 0)
$5A-B$	1722	2	True	True	3324	(861, 861, 0)
$7A-C$	1640	2	True	True	3238	(820, 820, 0)
$10A-B$	1722	2	True	True	3324	(861, 861, 0)
$20A-D$	1722	2	True	True	3324	(861, 861, 0)
$21A-F$	1640	2	True	True	3238	(820, 820, 0)
$41A-B$	840	20	True	True	2040	(420, 420, 0)

$PSL(2,43)$, order 39732

$\mathcal{C}$	$\|\mathcal{C}\|$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	903	23	True	True	1849	(903, 0, 0)
$3A$	1892	2	True	True	3658	(946, 946, 0)
$7A-C$	1892	2	True	True	3658	(946, 946, 0)
$11A-E$	1806	2	True	True	3568	(903, 903, 0)
$21A-F$	1892	2	True	True	3658	(946, 946, 0)
$22A-E$	1806	2	True	True	3568	(903, 903, 0)
$43A-B$	924	21	False	True	1344	(484, 440, 0)

$PSL(2,47)$, order 51888

$\mathcal{C}$	$\|\mathcal{C}\|$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	1081	25	True	True	2209	(1081, 0, 0)
$3A$	2162	2	True	True	4276	(1081, 1081, 0)
$4A$	2162	2	True	True	4276	(1081, 1081, 0)
$6A$	2162	2	True	True	4276	(1081, 1081, 0)
$8A-B$	2162	2	True	True	4276	(1081, 1081, 0)
$12A-B$	2162	2	True	True	4276	(1081, 1081, 0)
$23A-K$	2256	2	True	True	4374	(1128, 1128, 0)
$24A-D$	2162	2	True	True	4276	(1081, 1081, 0)
$47A-B$	1104	23	False	True	1610	(576, 528, 0)

$PSL(2,49)$, order 58800

$\mathcal{C}$	$\lvert\mathcal{C}\rvert$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	1225	25	True	True	2353	(1225, 0, 0)
$3A$	2450	2	True	True	4756	(1225, 1225, 0)
$4A$	2450	2	True	True	4756	(1225, 1225, 0)
$5A-B$	2352	2	True	True	4654	(1176, 1176, 0)
$6A$	2450	2	True	True	4756	(1225, 1225, 0)
$7A-B$	1200	24	True	True	2928	(600, 600, 0)
$8A-B$	2450	2	True	True	4756	(1225, 1225, 0)
$12A-B$	2450	2	True	True	4756	(1225, 1225, 0)
$24A-D$	2450	2	True	True	4756	(1225, 1225, 0)
$25A-J$	2352	2	True	True	4654	(1176, 1176, 0)

$PSU(3,4)$, order 62400

$\mathcal{C}$	$\lvert\mathcal{C}\rvert$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
$2A$	195	3	True	False (65)	201	(195, 0, 0)
$3A$	4160	2	True	True	8134	(2080, 2080, 0)
$4A$	3900	12	True	True	7824	(1950, 1950, 0)
$5A-D$	208	13	False	True	484	(79, 129, 0)
$5E-F$	2496	6	True	True	5436	(1248, 1248, 0)
$10A-D$	3120	3	False	True	3756	(1586, 1534, 0)
$13A-D$	4800	3	False	True	3948	(2310, 2490, 0)
$15A-D$	4160	2	False	True	4054	(2041, 2119, 0)

$PSL(2,53)$, order 74412

$\mathcal{C}$	$\lvert\mathcal{C}\rvert$	$\chi_{\mathcal{C}}(\mathcal{C})$	Real	Irred	$\lambda_{\max}$	Signature
2	1431	27	True	True	2757	(1431, 0, 0)
3	2756	2	True	True	5458	(1378, 1378, 0)
$9A-C$	2756	2	True	True	5458	(1378, 1378, 0)
$13A-F$	2862	2	True	True	5568	(1431, 1431, 0)
$26A-F$	2862	2	True	True	5568	(1431, 1431, 0)
$27A-I$	2756	2	True	True	5458	(1378, 1378, 0)
$53A-B$	1404	26	True	True	3432	(702, 702, 0)

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